

Fundamentals of Analog & Mixed Signal VLSI Design

Noise in Circuits and Systems

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Outline

- **Introduction**
- Random signals and noise
- Main noise sources of circuit components
- Noise models of basic components
- Noise calculation in continuous-time (CT) circuits
- Summary

Different Types of Noise – Interference Noise

- Unwanted interaction between circuit and noise sources
- Noise sources not always easy to identify and localize
- Can be **random** or **deterministic**
- Examples: power supply noise, capacitive coupling, substrate coupling
- Can be **reduced** by **careful wiring, layout and grounding**
- Interference noise will **not be covered here**



David Johns and Ken Martin, *Analog Integrated Circuit Design*, 1st ed., Wiley 1997.

Different Types of Noise – Inherent Noise

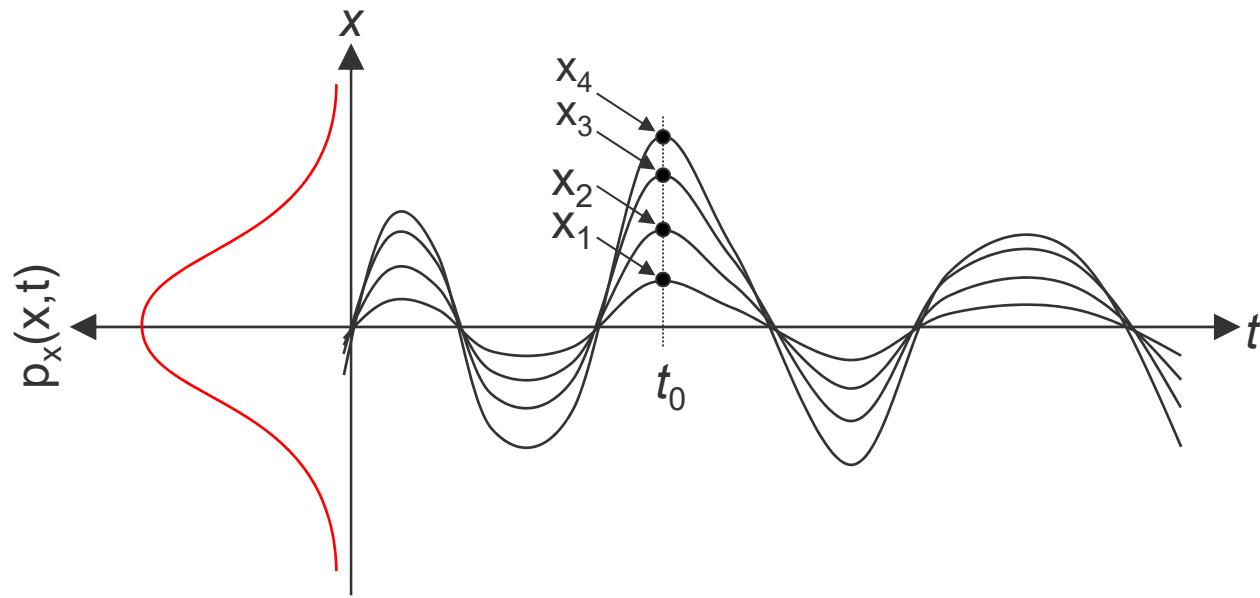
- Noise generated in **active** and **passive devices** (mainly resistors)
- **Random noise** due to **fluctuations** of physical quantities such as the carrier velocity
- Since source is random → it can be reduced but **NEVER eliminated**
- Examples: **thermal noise**, **shot noise**, and **flicker noise**
- Not strongly affected by wiring or layout
- Noise can be reduced by proper circuit design
- To design low-noise circuits requires understanding the parameters that determine the noise sources and how these noise sources propagate in the circuit
- Requires tools for analyzing and optimizing noise



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Random Process – Definition



- The **fluctuation** of physical quantities can be mathematically modeled by a **random** or **stochastic process**
- A random process or stochastic process can be defined as a family of function $x(t)$, where $x(t = t_0)$ is a **random variable** which has a 1st-order distribution function $p_x(x, t)$
- The instantaneous amplitude of the signal $x(t)$ at time t_0 **cannot be predicted** since it is a random variable

Random Process – Mean Value, Autocorrelation

- The **mean** or **expected value** $m_x(t)$ is obtained by averaging the amplitude x at a given time t over all possible values

$$m_x(t) = E[x(t)] \triangleq \int_{-\infty}^{+\infty} x \cdot p_x(x, t) \cdot dx$$

- The **autocorrelation** function (ACF) evaluates the statistical dependency between two instantaneous values at time t_1 and t_2

$$R_x(t_1, t_2) = E[x(t_1)x(t_2)] \triangleq \iint_{-\infty}^{+\infty} x_1 \cdot x_2 \cdot p_x(x_1, x_2, t_1, t_2) \cdot dx_1 dx_2$$

- where $x_1 = x(t_1)$, $x_2 = x(t_2)$ and $p_x(x_1, x_2, t_1, t_2)$ is the 2nd-order distribution function of process $x(t)$
- The **autocovariance** function is defined by

$$C_x(t_1, t_2) = E[(x(t_1) - m_x(t_1))(x(t_2) - m_x(t_2))] = R_x(t_1, t_2) - m_x(t_1) \cdot m_x(t_2)$$

Random Process – Stationary Process

- The random process is called **stationary** in the wide sense if its **mean value is constant** and its autocorrelation function only depends on the **time difference**

$$\tau = t_2 - t_1$$

$$m_x = \text{constant}$$

$$R_x(t_1, t_2) = R_x(\tau)$$

$$C_x(t_1, t_2) = C_x(\tau) = R_x(\tau) - m_x^2$$

- Note that $C_x(\tau) = R_x(\tau)$ for $m_x = 0$
- The normalized autocovariance function (or correlation coefficient) is defined as

$$\rho_x(\tau) = \frac{C_x(\tau)}{\sigma_x^2} \text{ where } \sigma_x^2 = C_x(0) \text{ is the } \text{variance}$$

- For **real** processes $R_x(\tau)$ and $C_x(\tau)$ are **even functions**

$$R_x(\tau) = R_x(-\tau) \text{ and } C_x(\tau) = C_x(-\tau)$$

- and have the following properties

$$C_x(0) = \sigma_x^2, R_x(0) = \sigma_x^2 - m_x^2 \text{ and } \rho_x(0) = 1$$



Random Process – Ergodic Process

- A process is said **ergodic** when time average is equal to ensemble average

$$\bar{x} = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_0^T x(t) \cdot dt = E[x] = m_x$$

$$\overline{x^2} = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_0^T x^2(t) \cdot dt = E[x^2] = R_x(0)$$



A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

Random Process – Power Spectral Density (PSD)

- Noise corresponding to a **stationary** process $x(t)$ is often characterized in the frequency domain by the **Power Spectral Density** (PSD) which is the Fourier transform of its autocorrelation function

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} = \int_{-\infty}^{+\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_x(\tau) = \mathcal{F}^{-1}\{S_x(f)\} = \int_{-\infty}^{+\infty} S_x(f) e^{+j2\pi f\tau} df$$

- PSD unit in [V²/Hz] if $x(t)$ is a voltage or [A²/Hz] if $x(t)$ is a current
- The above definition use the **bilateral Fourier transform (so watch out factor 2)!**
- Usually the unilateral PSD $S_x^+(f)$ is measured or simulated. It is related to the bilateral Fourier transform by

$$S_x^+(f) = \begin{cases} 0 & \text{for } f < 0 \\ S_x(f) & \text{for } f = 0 \\ 2S_x(f) & \text{for } f > 0 \end{cases}$$

Random Process – Power and RMS Value

- The **total power** associated to the random process $x(t)$ is simply obtained by integrating its PSD over frequency

$$P_x = \int_{-\infty}^{+\infty} S_x(f) df = R_x(0)$$

- The total power corresponds to the square of the RMS value
- For a noise **voltage** PSD

$$V_{n,rms}^2 = \int_{-\infty}^{+\infty} S_{v_n}(f) df = \int_0^{+\infty} S_{v_n}^+(f) df$$

- For a noise **current** PSD

$$I_{n,rms}^2 = \int_{-\infty}^{+\infty} S_{i_n}(f) df = \int_0^{+\infty} S_{i_n}^+(f) df$$



Random Process – Cross-correlation and Cross-PSD

- If we have two real jointly stationary processes $x(t)$ and $y(t)$, we can define their cross-correlation function (CCF) and cross-covariance by

$$R_{xy}(\tau) = E[x(t + \tau)y(t)]$$

$$C_{xy}(\tau) = E[(x(t + \tau) - m_x)(y(t + \tau) - m_y)] = R_{xy}(\tau) - m_x \cdot m_y$$

- The normalized cross-covariance is defined as

$$\rho_{xy}(\tau) = \frac{C_{xy}(\tau)}{\sigma_x \cdot \sigma_y}$$

- Two processes $x(t)$ and $y(t)$ are **uncorrelated** if $C_{xy}(\tau) = 0$
- The cross-PSD of the two real jointly stationary processes $x(t)$ and $y(t)$, is the Fourier transform of its cross-correlation function

$$S_{xy}(f) = \mathcal{F}\{R_{xy}(\tau)\} = \int_{-\infty}^{+\infty} R_{xy}(\tau) e^{-j2\pi f\tau} d\tau$$

Noise Summation and Correlation

- If $z(t) = x(t) + y(t)$ is the sum of two **real stationary processes**, then

$$R_z(\tau) = R_x(\tau) + R_y(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$$

- In case processes $x(t)$ and $y(t)$ are **uncorrelated**, then

$$C_{xy}(\tau) = C_{yx}(\tau) = 0 \Rightarrow R_{xy}(\tau) = R_{yx}(\tau) = m_x m_y \Rightarrow R_z(\tau) = R_x(\tau) + R_y(\tau) + 2m_x m_y$$

- If in addition $m_x = 0$ or $m_y = 0$ (or both), then

$$R_z(\tau) = R_x(\tau) + R_y(\tau)$$

- In this case the variance of $z(t)$ is given by

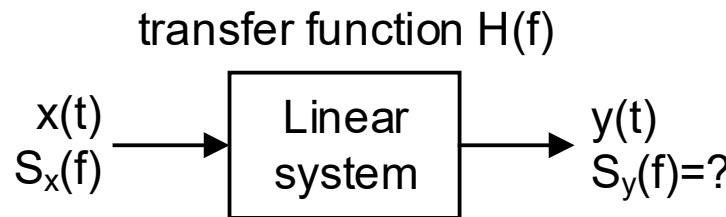
$$R_z(0) = \sigma_z^2 = R_x(0) + R_y(0) = \sigma_x^2 + \sigma_y^2$$

- In case process $x(t)$ and $y(t)$ are **correlated** and $m_x = 0$ or $m_y = 0$ (or both), then the variance is given by

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 + 2c\sigma_x\sigma_y \text{ where } c = \rho_{xy}(0) = \rho_{yx}(0)$$

- where c is the **correlation coefficient**

PSD at the Output of a Linear System

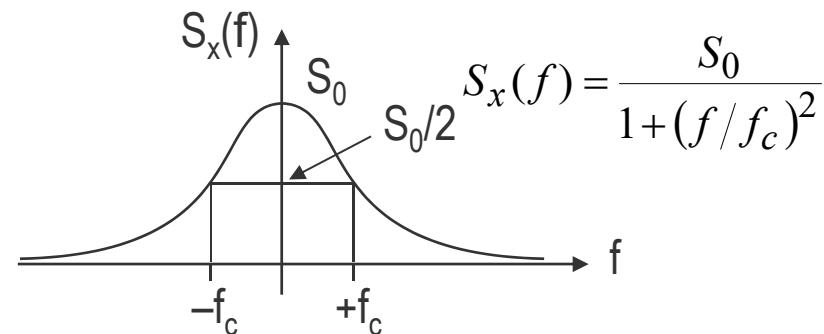
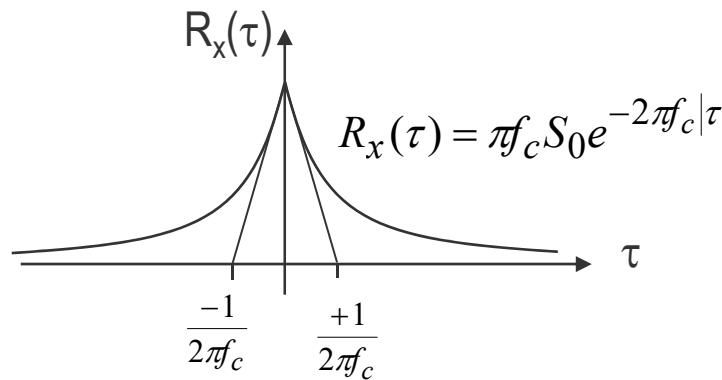
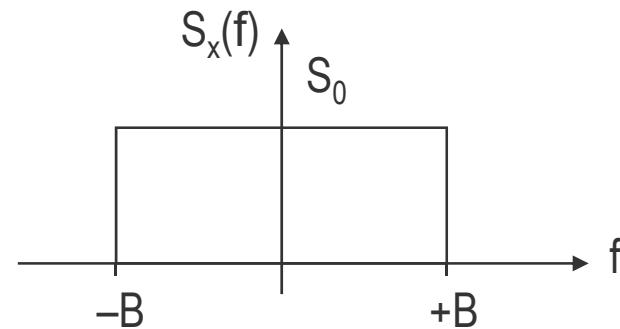
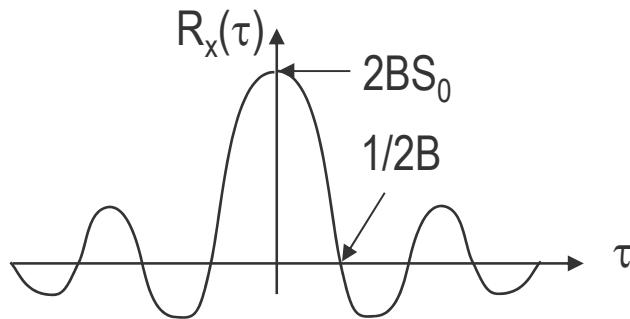
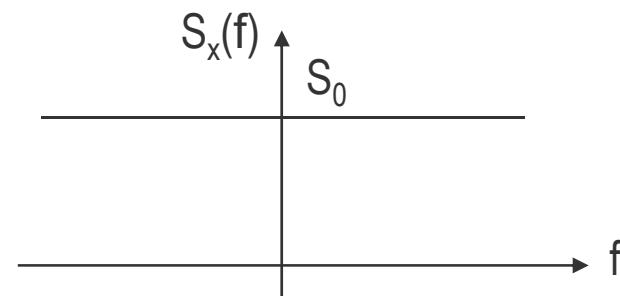
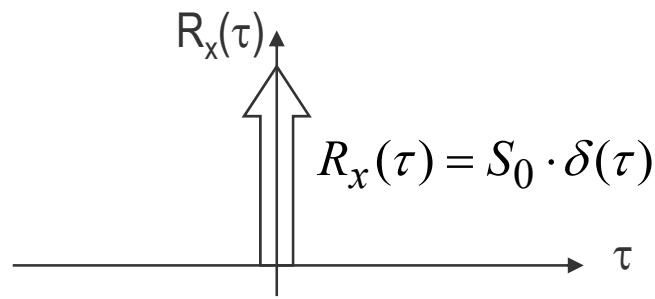


- Assuming the input is a **stationary random process** $x(t)$ having a PSD $S_x(f)$, the output $y(t)$ will also be a stationary random process having a PSD $S_y(f)$ given by

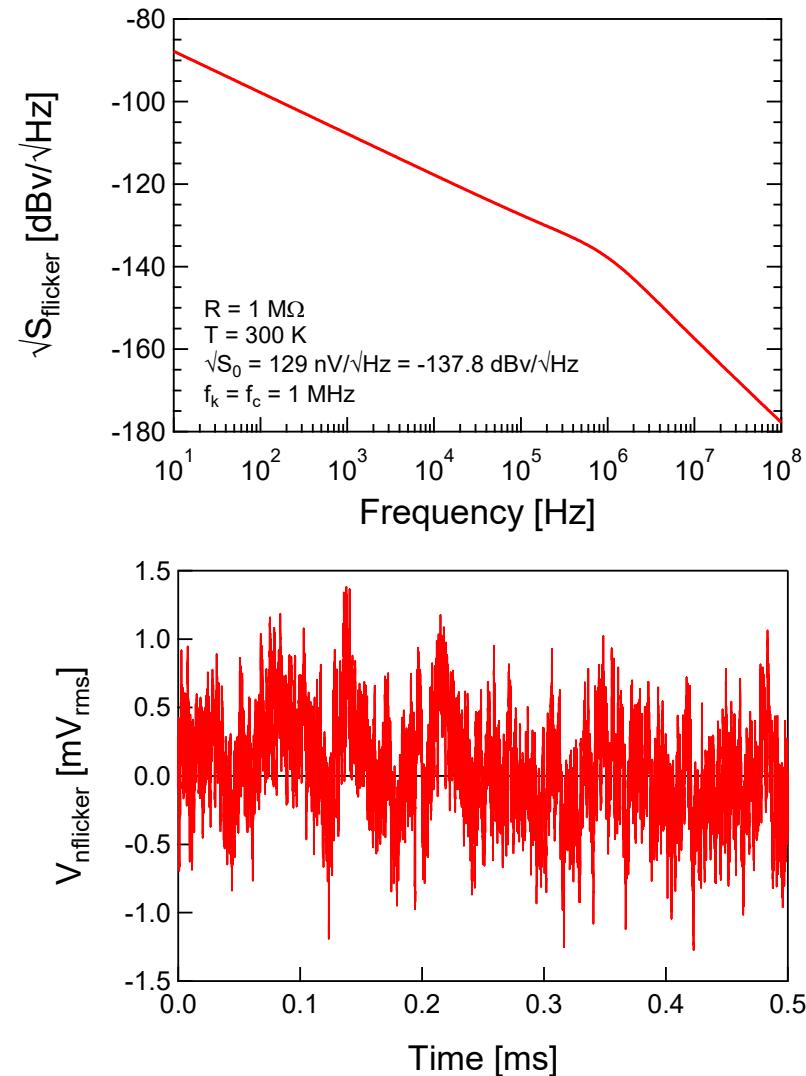
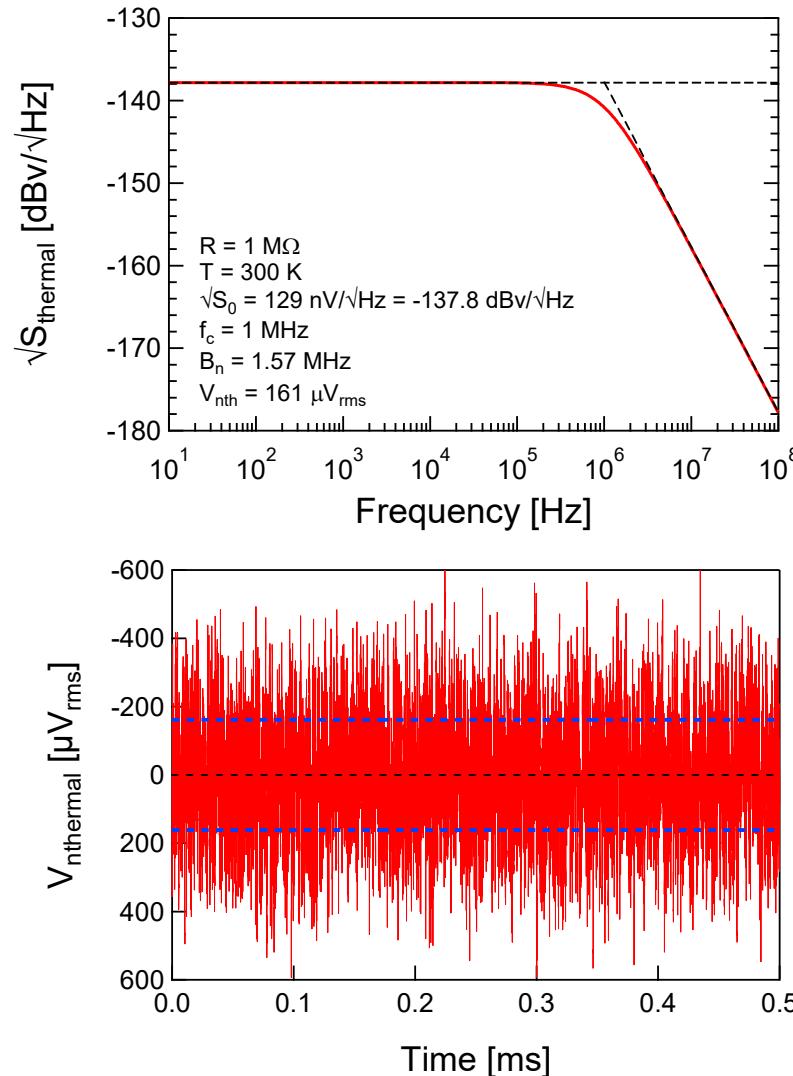
$$S_y(f) = |H(f)|^2 \cdot S_x(f)$$

- The above relation is called the **Wiener-Kintchine theorem**
- The linear system could be a filter which shapes the input noise according to its square magnitude (phase does not play a role in this case)
- If the input is a white noise $S_x(f) = S_0 = \text{cte}$, the output PSD is then proportional to the square magnitude of the filter transfer function

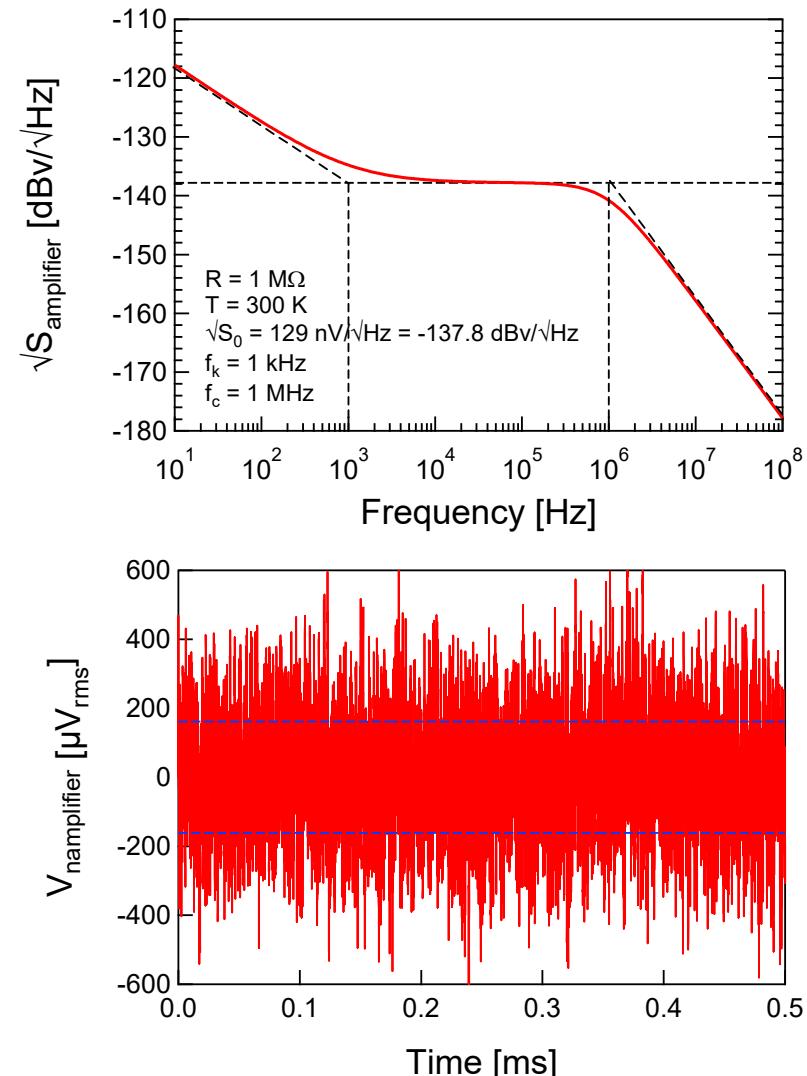
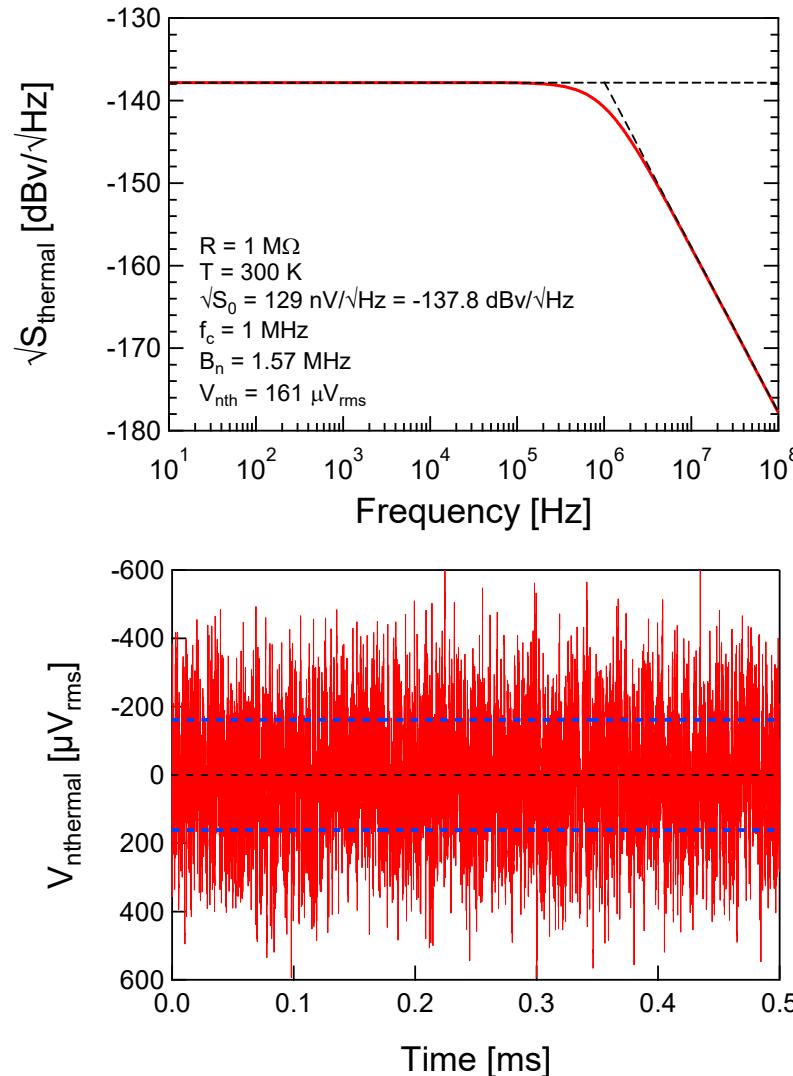
White Noise



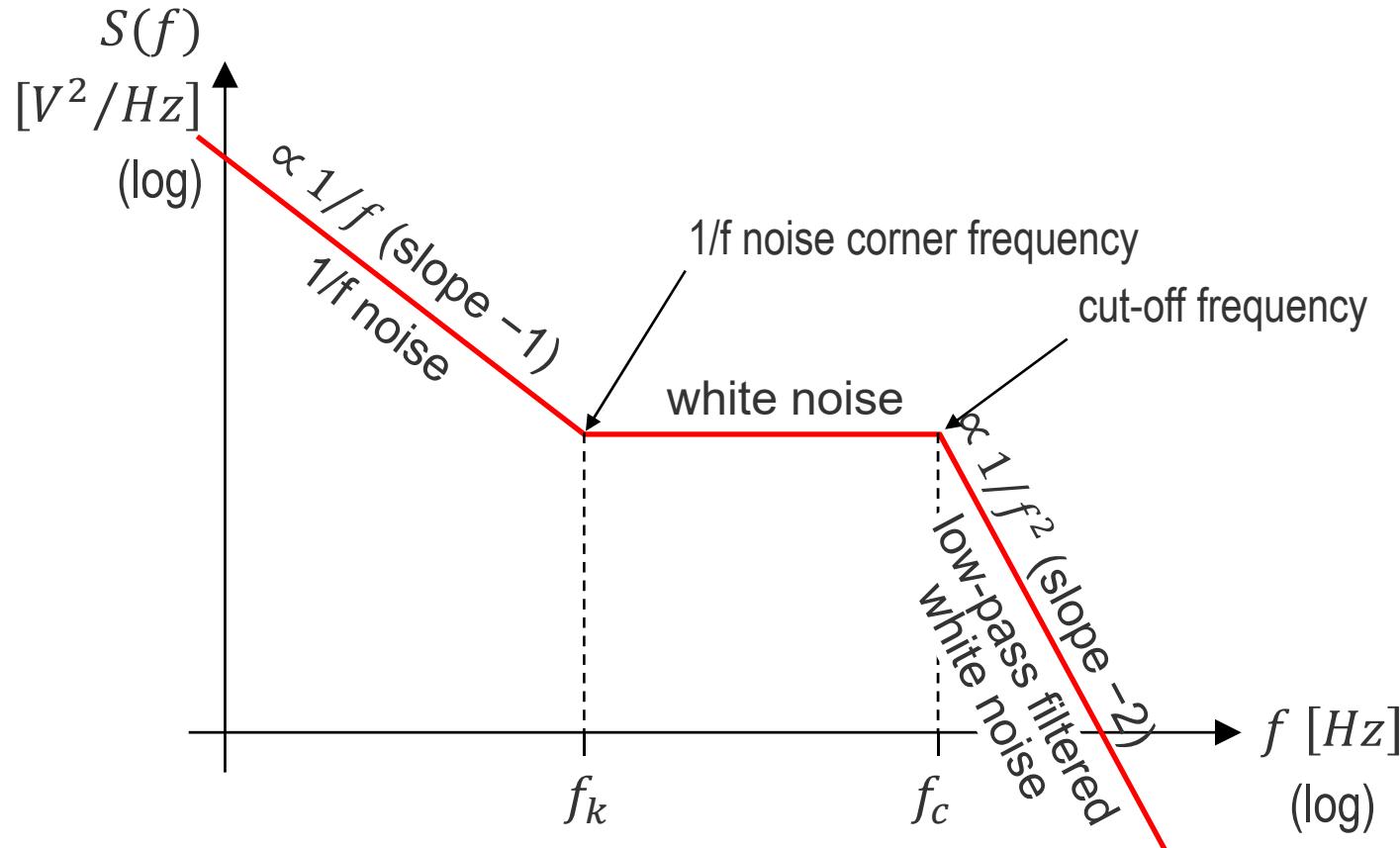
Noise in Frequency and Time Domain (1/2)



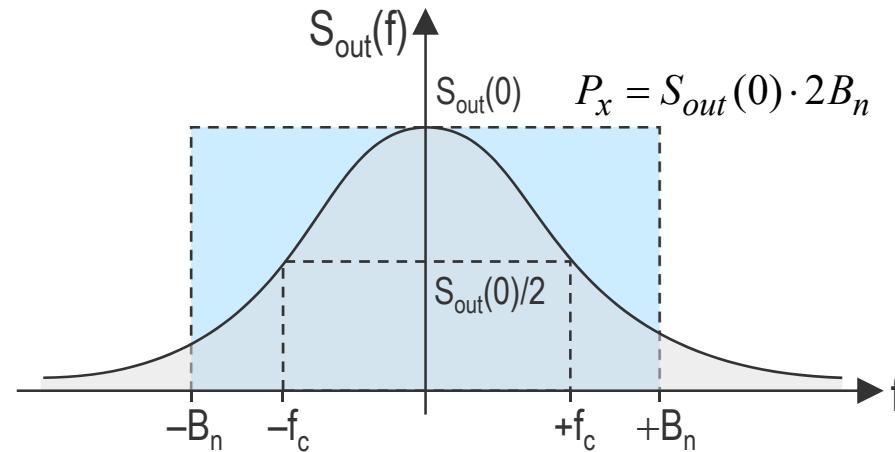
Noise in Frequency and Time Domain (2/2)



Typical PSD Shape (output-referred)



Equivalent Noise Bandwidth



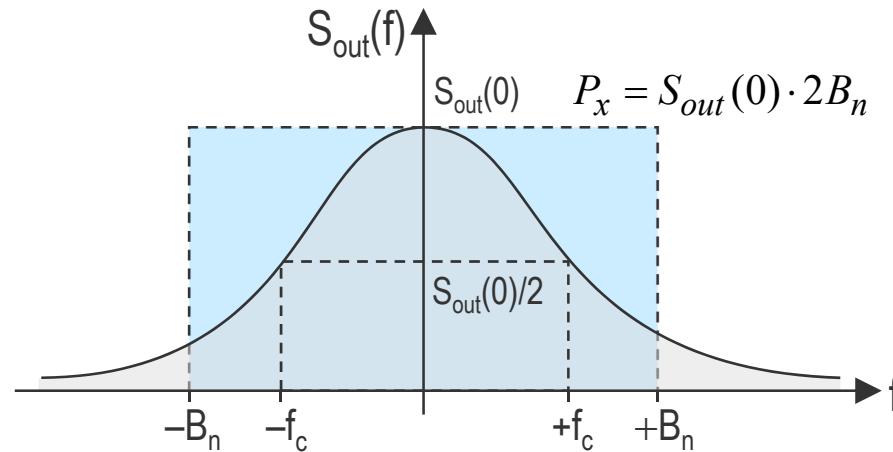
- The **equivalent noise bandwidth** of a low-pass noise PSD is defined as the bandwidth B_n of an ideally low-pass filtered **white noise** having the same value $S_{out}(0)$ (at $f = 0$) and the same power P_x than the original noise

$$B_n = \frac{1}{2} \cdot \frac{1}{S_{out}(0)} \cdot \int_{-\infty}^{+\infty} S_{out}(f) \cdot df = \frac{P_{out}}{2S_{out}(0)}$$

- If the input noise is a white noise of PSD S_0 , the output noise is then given by $S_{out}(f) = |H(f)|^2 \cdot S_0$ and the **equivalent bandwidth only depends on the transfer function**

$$B_n = \frac{1}{2} \cdot \frac{1}{|H(0)|^2 S_0} \cdot \int_{-\infty}^{+\infty} |H(f)|^2 \cdot S_0 \cdot df = \frac{1}{|H(0)|^2} \cdot \int_0^{+\infty} |H(f)|^2 \cdot df$$

Equivalent Noise Bandwidth of a 1st-order LP Filtered White Noise



- The equivalent noise bandwidth of a **1st-order low-pass filtered white noise** with a cut-off frequency f_c is given by

$$B_n = \int_0^{+\infty} \frac{df}{1 + (f/f_c)^2} = f_c \cdot \underbrace{\int_0^{+\infty} \frac{dx}{1 + x^2}}_{=\frac{\pi}{2}} = \frac{\pi}{2} f_c$$

Noise Bandwidth of Various Filters

Type	Noise Transfer Function	Noise Bandwidth
1 st -order LP	$\frac{1}{1 + \frac{s}{\omega_c}}$	$\frac{\omega_c}{4}$
2 nd -order LP	$\frac{1}{1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2}$	$\frac{\omega_0 \cdot Q}{4}$
2 nd -order BP ¹	$\frac{\frac{s}{\omega_0 \cdot Q}}{1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2}$	$\frac{\omega_0}{4Q}$
2 nd -order LP (with zero)	$\frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2}$	$\frac{\omega_0 \cdot Q}{4} \cdot \left[1 + \left(\frac{\omega_0}{\omega_z}\right)^2 \right]$

1. In the case of the bandpass filter the dc gain $H_0(0)$ is zero and should be replaced in the above definition by the gain at the resonance frequency which in the above case has been set to one.

Higher Order Transfer Functions

3rd-order LP

- Noise transfer function:
- Noise bandwidth:

$$H_{n3}(s) = \frac{n_2 s^2 + n_1 s + n_0}{d_3 s^3 + d_2 s^2 + d_1 s + d_0}$$

$$B_{n3} = \frac{1}{4} \frac{n_2^2 d_1 d_0 + n_1^2 d_3 d_0 + n_0^2 d_3 d_2 - 2n_2 n_0 d_3 d_0}{d_3 (d_2 d_1 - d_3 d_0) d_0}$$

4th-order LP

- Noise transfer function:
- Noise bandwidth:

$$H_{n4}(s) = \frac{n_3 s^3 + n_2 s^2 + n_1 s + n_0}{d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0}$$

$$B_{n4} = \frac{1}{4} \frac{n_3^2 d_3 d_0^2 + 2n_3 n_1 d_4 d_1 d_0 - n_3^2 d_2 d_1 d_0 - n_2^2 d_4 d_1 d_0 - n_1^2 d_4 d_3 d_0 - n_0^2 d_4 d_3 d_2 + 2n_2 n_0 d_4 d_3 d_0 + n_0^2 d_4^2 d_1}{d_4 (d_4 d_1^2 - d_3 d_2 d_1 + d_3^2 d_0) d_0}$$

Contribution of 1/f Noise to the Total Noise Power

- The PSD (single-sided) of an amplifier including the 1/f noise component can be written as

$$S_n(f) = S_0 \cdot \left(1 + \frac{f_k}{|f|} \right)$$

- where S_0 is the white noise component and f_k the corner frequency (frequency at which the 1/f noise becomes equal to the white noise)
- The noise power assuming the noise is filtered by a 1st-order low-pass filter having a cut-off frequency f_c is the given by

$$V_n^2 = \int_0^{+\infty} \frac{S_n(f)}{1 + (f/f_c)^2} \cdot df$$

- We then can define an equivalent noise bandwidth B_n including 1/f noise defined as

$$B_n \triangleq \frac{V_n^2}{S_0} = \int_0^{+\infty} \frac{1 + f_k/f}{1 + (f/f_c)^2} \cdot df = f_c \cdot \int_0^{+\infty} \frac{dx}{1 + x^2} + f_k \cdot \int_{x_\ell}^{+\infty} \frac{dx}{x \cdot (1 + x^2)}$$

- where $x \triangleq f/f_c$ and $x_\ell \triangleq f_\ell/f_c \ll 1$. It is convenient to choose $f_\ell = 1 \text{ Hz}$

$$B_n = \frac{\pi}{2} f_c + \frac{f_k}{2} \ln \left[1 + \left(\frac{f_c}{f_\ell} \right)^2 \right] \cong \frac{\pi}{2} f_c + f_k \ln \left(\frac{f_c}{f_\ell} \right) = \frac{\pi}{2} f_c + f_k \ln \left(\frac{f_c}{1 \text{ Hz}} \right)$$

- We see that the contribution of 1/f noise to the total noise power is scaling only with the $\ln f_c$ whereas the white noise scales proportionally to f_c

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Main Noise Sources of Circuit Components

Three main sources of noise:

- **Thermal Noise**

- ▶ Due to thermal excitation of charge carriers
 - ▶ Appears as white spectral density

- **Shot Noise**

- ▶ Due to carriers randomly crossing a barrier
 - ▶ Depends on dc bias current and is white

- **Flicker Noise**

- ▶ Due to traps in semiconductors
 - ▶ Has a $1/f$ spectral density
 - ▶ Significant in MOS transistors at low frequencies

Thermal Noise – The Equipartition Theorem

- Every closed physical system at temperature T contains energy of average amount $kT/2$ per degree of freedom, where $k = 1.38 \cdot 10^{-23} \text{ J/K}$ is the Boltzmann constant
- Consider a gas of electrons having a Maxwellian velocity distribution

$$p(v) = \sqrt{\frac{m}{2\pi kT}} \cdot e^{-\frac{mv^2}{2kT}}$$

where $p(v) \cdot dv$ represents the probability of finding one electron having a velocity comprised between v and $v + dv$

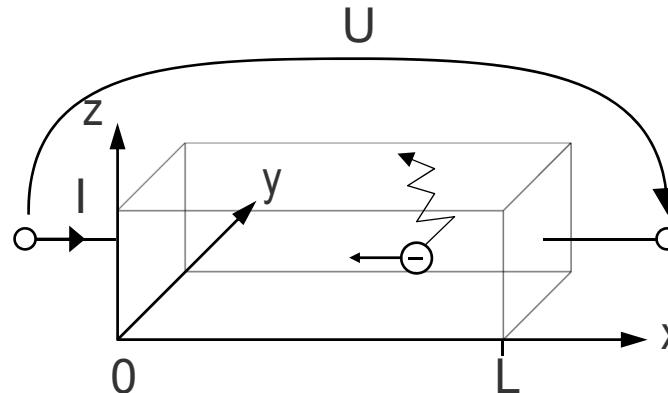
- The average energy of the electron at equilibrium is given by

$$\overline{W} = E\left[m\frac{v^2}{2}\right] = \frac{m}{2} E[v^2] = \frac{m}{2} \cdot \int_0^{+\infty} v^2 \cdot p(v) \cdot dv = \frac{kT}{2}$$

- Hence, for a one degree of freedom system, we have

$$\frac{mv^2}{2} = \frac{kT}{2}$$

The Nyquist Theorem – Microscopic Derivation



A : section
 n : electron density

- The voltage is given by Ohm's law

$$U = R \cdot I = R \cdot A \cdot J = R \cdot A \cdot n \cdot q \cdot v$$

- where A is the section, n the electron density and v is the **drift velocity** along the x axis, averaged over the ensemble of electrons N

$$v = \frac{1}{N} \cdot \sum_i v_i \text{ with } N = n \cdot A \cdot L$$

- The voltage U is then given by

$$U = \frac{q \cdot R}{L} \cdot \sum_i v_i = \sum_i u_i \text{ where } u_i = \frac{q \cdot R}{L} \cdot v_i$$



The Nyquist Theorem – Microscopic Derivation

- If no dc current flows, then $I = 0$, $U = 0$ and $v = 0$
- On the other hand the variances are non-zero

$$\overline{v^2} = E[v^2] \neq 0 \text{ and } \overline{U^2} = E[U^2] \neq 0$$

- Assuming no correlation between electrons, the ACF of U is given by

$$R_u(\tau) = \sum_i R_{u_i}(\tau) = \left(\frac{q \cdot R}{L}\right)^2 \cdot \sum_i R_{v_i}(\tau) \text{ where } R_{v_i}(\tau) \text{ is the ACF of } v_i$$

- If v_i has a Maxwellian distribution then $R_{v_i}(\tau)$ is given by

$$R_{v_i}(\tau) = R_{v_i}(0) \cdot e^{-\frac{|\tau|}{\tau_0}}$$

- where τ_0 is the **relaxation time** or mean time of flight of the conduction electrons
- $R_{v_i}(0)$ can be found using the equipartition theorem according to

$$m \frac{v_i^2}{2} = \frac{m}{2} R_{v_i}(0) = \frac{kT}{2} \text{ which results in } R_{v_i}(0) = \frac{kT}{m}$$

- The ACF of the voltage is then simply given by

$$R_u(\tau) = N \cdot \left(\frac{q \cdot R}{L}\right)^2 \cdot \frac{kT}{m} \cdot e^{-\frac{|\tau|}{\tau_0}}$$

The Nyquist Theorem – Microscopic Derivation

- The corresponding bilateral PSD of the noise voltage is a **Lorentzian spectrum**

$$S_u(f) = N \cdot \left(\frac{q \cdot R}{L} \right)^2 \cdot \frac{kT}{m} \cdot \frac{2\tau_0}{1 + (2\pi f \tau_0)^2}$$

- Usually in metals at room temperature $\tau_0 < 10^{-13} s$ and hence $2\pi f \tau_0 \ll 1$

$$S_u(f) \cong N \cdot \left(\frac{q \cdot R}{L} \right)^2 \cdot \frac{kT}{m} \cdot 2\tau_0 = 2kTR \cdot \frac{n \cdot A \cdot q^2 \cdot R \cdot \tau_0}{m \cdot L} \text{ for } f \ll \frac{1}{2\pi\tau_0}$$

- Recalling that the conductivity σ is given by $\sigma = q\mu n$ and the mobility μ by $\mu = q\tau_0/m$, the resistance can then be written as

$$R = \frac{L}{\sigma \cdot A} = \frac{L}{A} \cdot \frac{m}{q^2 \cdot n \cdot \tau_0}$$

- Replacing R in the above right term leads to the **bilateral** PSD of thermal noise

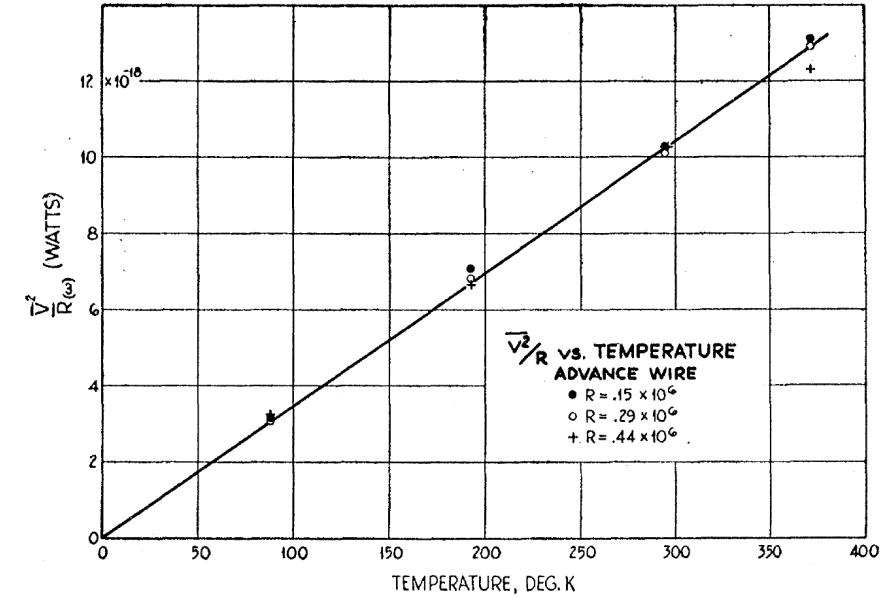
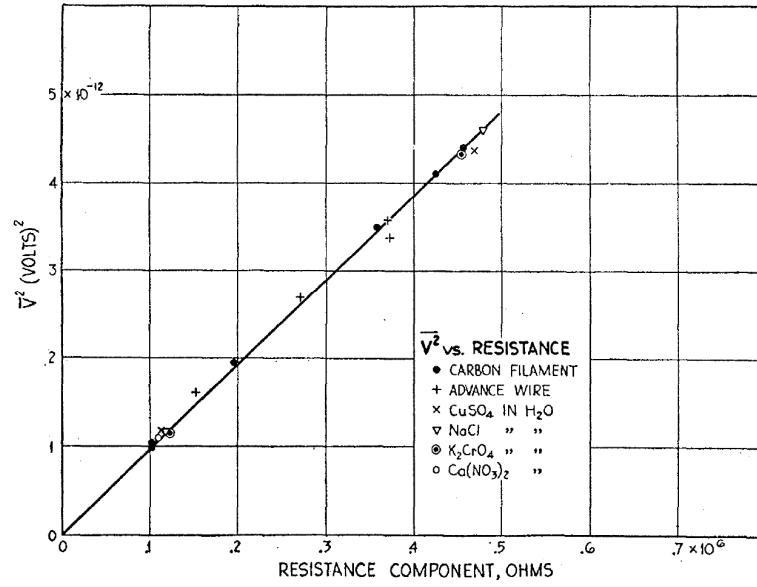
$$S_u(f) = 2kTR$$

- or in **unilateral** form

$$S_u(f) = 4kTR$$

Thermal Noise – The Original Measurements by Johnson

$$S_{th} = 4kT \cdot R$$

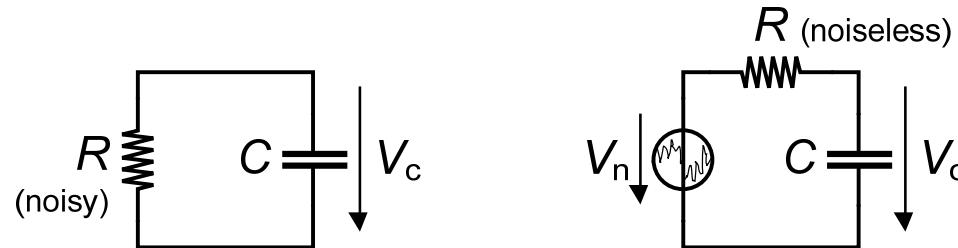


- The thermal noise PSD is
 - proportional to the resistance R (left figure)
 - proportional to temperature T (right figure)
 - independent of the current flowing through the resistor



J. B. Johnson, "Thermal Agitation of Electricity in Conductors," Physical Review, vol. 32, pp. 97-109, July 1928.

Thermal Noise – kT/C Noise



- The **variance** of the thermal noise voltage V_c across C is given by

$$\overline{V_c^2} = 4kTR \cdot \int_0^{+\infty} \frac{df}{1 + (2\pi f \tau)^2} = 4kTR \cdot B_n = 4kTR \cdot \frac{\pi}{2} \frac{1}{2\pi RC} = \frac{kT}{C}$$

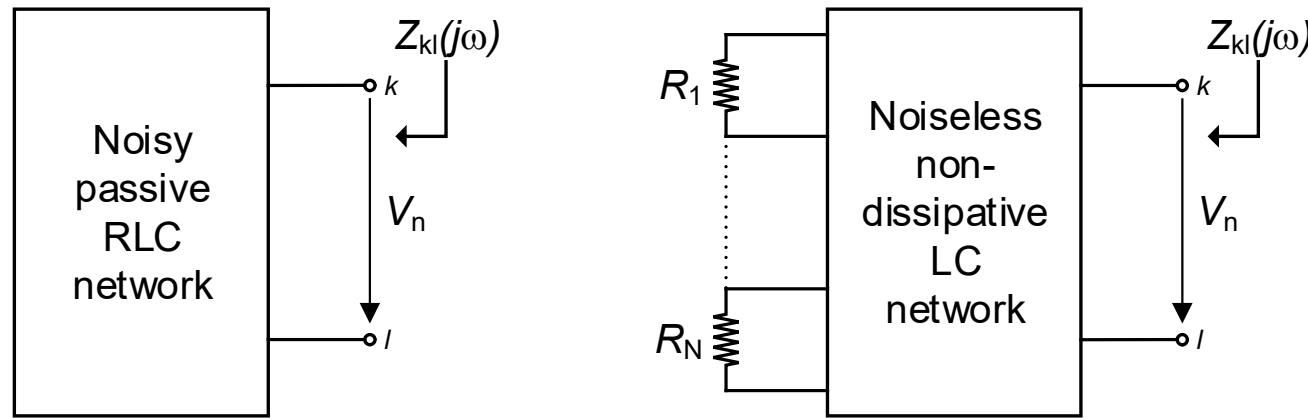
- This result can be obtained directly by applying the **equipartition theorem**
- Average stored energy in C is given by $\bar{W} = \frac{1}{2} C \cdot \overline{V_c^2}$
- Since resistor R and capacitor C are in thermal equilibrium and there is only one degree of freedom, we have

$$\bar{W} = kT/2$$

- We then get

$$C \cdot \overline{V_c^2} = kT \Rightarrow \overline{V_c^2} = \frac{kT}{C}$$

Thermal Noise in Passive Networks – The Nyquist Theorem



- The power spectral density (PSD) of noise voltage V_n is given by

$$S_{v_n}(f) = 4kT \cdot \Re\{Z_{kl}(j2\pi f)\}$$

- The variance of voltage V_n is then given by

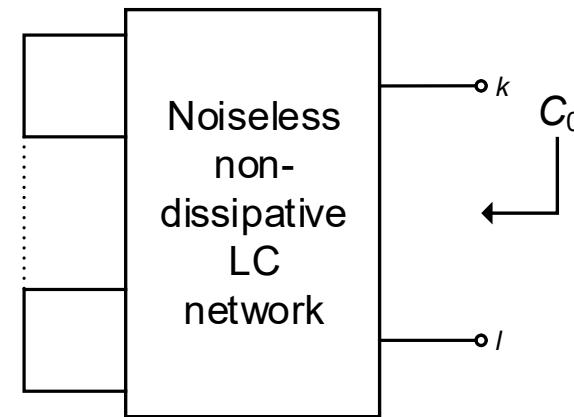
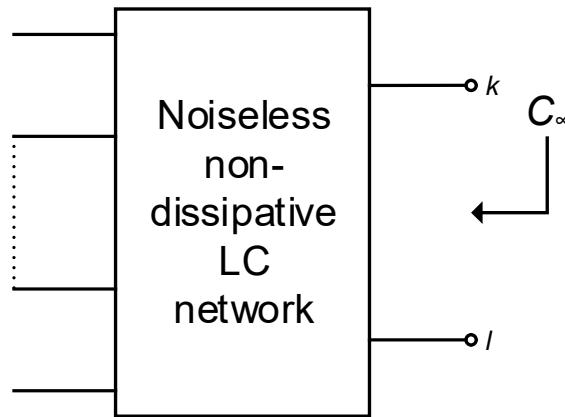
$$\overline{V_n^2} = \int_0^{+\infty} S_{v_n}(f) \cdot df = 4kT \cdot \int_0^{+\infty} \Re\{Z_{kl}(j2\pi f)\} \cdot df$$

- Or we can use the Bode theorem given in the next slide

 H. Nyquist, "Thermal Agitation of Electric Charge in Conductors," Physical Review B, vol. 32, pp. 110-113, July 1928.

 A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., pp. 362-363, 1981.

Thermal Noise in Passive Networks – The Bode Theorem



$$\frac{1}{C_{\infty}} = \lim_{s \rightarrow \infty} [s \cdot Z(s)]$$

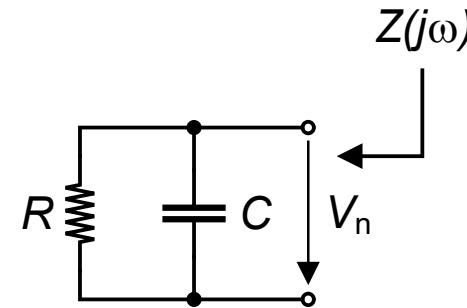
$$\frac{1}{C_0} = \lim_{s \rightarrow 0} [s \cdot Z(s)]$$

- The variance of noise voltage V_n can be obtained without computing the integral by using the Bode theorem stating

$$\overline{V_n^2} = kT \cdot \left[\frac{1}{C_{\infty}} - \frac{1}{C_0} \right]$$

- Where C_{∞} and C_0 are defined as follows

Nyquist and Bode Theorems – Example (1/2)



- The impedance is given by

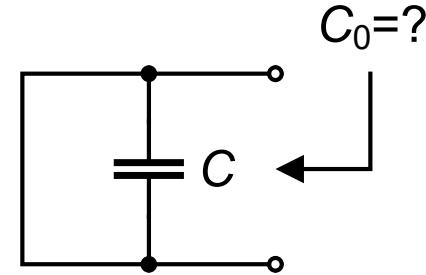
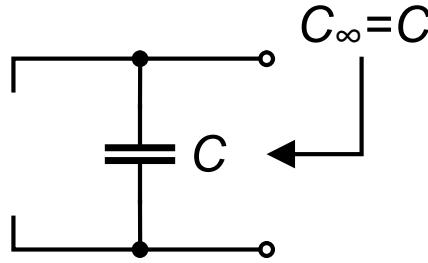
$$Z(j\omega) = \frac{1}{1/R + j\omega C} = \frac{R}{1 + j\omega RC} = \frac{R}{1 + (\omega RC)^2} - j \frac{\omega R^2 C}{1 + (\omega RC)^2}$$

- And hence $S_{v_n}(f) = 4kT \cdot \Re\{Z(j2\pi f)\} = \frac{4kTR}{1 + (\omega RC)^2}$

- The noise variance is then given by

$$\begin{aligned} \overline{V_n^2} &= \int_0^{+\infty} S_{v_n}(f) \cdot df = 4kTR \cdot \int_0^{+\infty} \frac{df}{1 + (2\pi f \tau)^2} = 4kTR \cdot B_n \\ &= 4kTR \cdot \frac{\pi}{2} \frac{1}{2\pi RC} = \frac{kT}{C} \end{aligned}$$

Nyquist and Bode Theorems – Example (2/2)



$$\frac{1}{C_\infty} = \lim_{s \rightarrow \infty} [s \cdot Z(s)] = \lim_{s \rightarrow \infty} \frac{sR}{1 + sRC} = \frac{1}{C} \quad \frac{1}{C_0} = \lim_{s \rightarrow 0} [s \cdot Z(s)] = \lim_{s \rightarrow 0} \frac{sR}{1 + sRC} = 0$$

- Noise variance of a 1st-order RC circuit using Bode theorem results in

$$\overline{V_n^2} = kT \cdot \left[\frac{1}{C_\infty} - \frac{1}{C_0} \right] = kT \cdot \left[\frac{1}{C} - 0 \right] = \frac{kT}{C}$$

- Just by circuit inspection without any integration!
- However, unfortunately only applies to passive circuits!

Shot Noise – The Poisson Process

- The Poisson process is characterized by a **sequence of independent random events**, occurring at any time t_k with the **same probability**

- The probability to have exactly n events in the time interval $[0, t]$ is given by

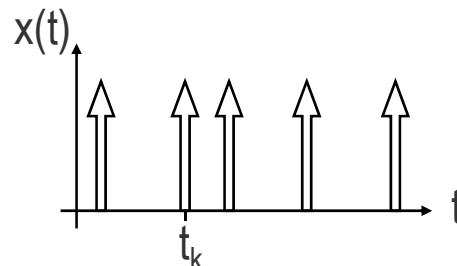
$$p_n(t) = \frac{(\lambda \cdot t)^n}{n!} \cdot e^{-\lambda \cdot t}$$

- where λ is the **average number of events per second**, which can be assumed to be constant
- The average (or expected) number of events therefore grows **linearly with time**

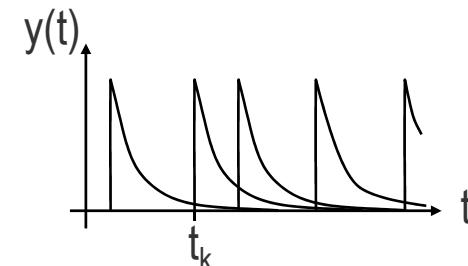
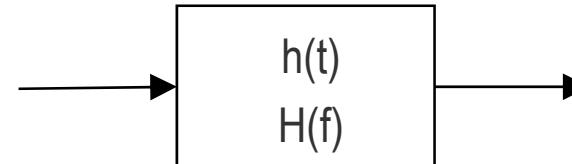
$$E[n] = \sum_{n=0}^{+\infty} n \cdot p_n(t) = \lambda \cdot t$$



Shot Noise – The Shot Noise Process

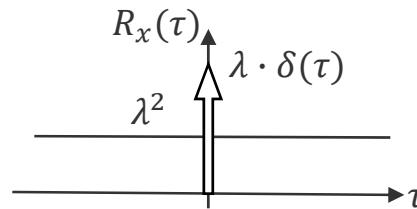


$$x(t) = \sum_k \delta(t - t_k)$$



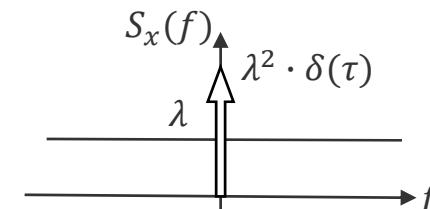
$$y(t) = \sum_k h(t - t_k)$$

- The **shot noise** process is defined by $y(t) = \sum_k h(t - t_k)$, where t_k are random points in time with uniform density λ
- $y(t)$ can be considered as the output of a linear system having an impulse response $h(t)$ and a sequence of **Poisson impulses** at the input $x(t) = \sum_k \delta(t - t_k)$
- It can be shown that the PSD is made of a **DC component $\lambda^2 \cdot \delta(f)$** and an additional **white noise λ**



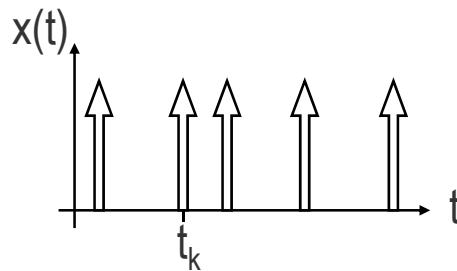
$$R_x(\tau) = \lambda^2 + \lambda \cdot \delta(\tau)$$

$$S_x(f) = \lambda^2 \cdot \delta(\tau) + \lambda$$

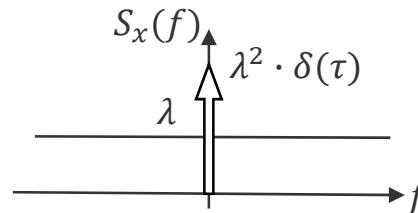
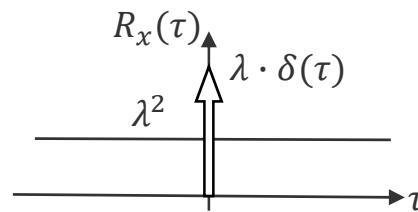


A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

Shot Noise – The Shot Noise Process

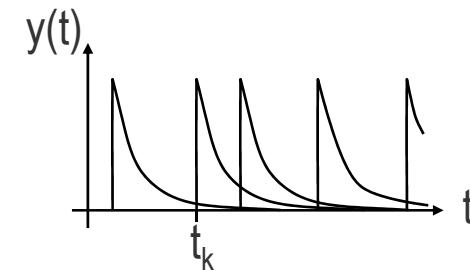
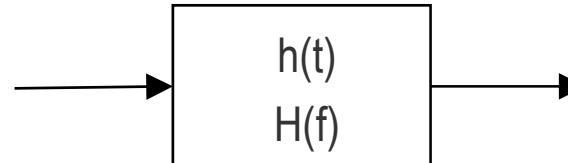


$$x(t) = \sum_k \delta(t - t_k)$$

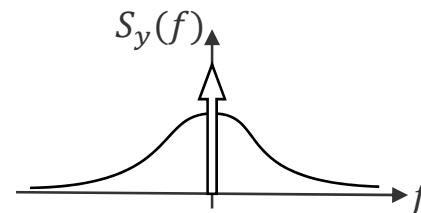
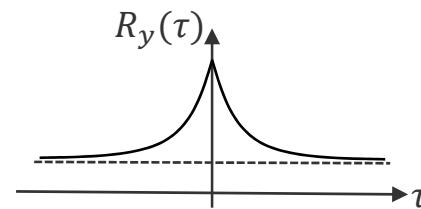


$$R_x(\tau) = \lambda^2 + \lambda \cdot \delta(\tau)$$

$$S_x(f) = \lambda^2 \cdot \delta(f) + \lambda$$



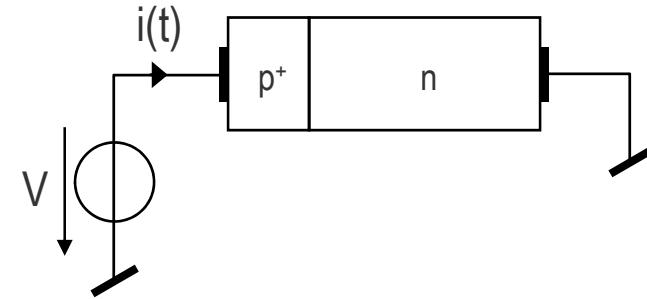
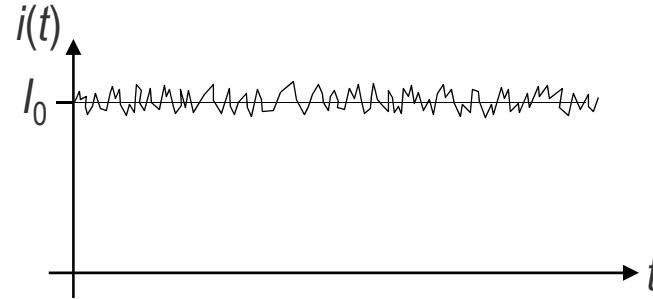
$$y(t) = \sum_k h(t - t_k)$$



$$R_y(\tau) = \lambda^2 \cdot |H(0)|^2 + \lambda \cdot \int_{-\infty}^{+\infty} h(t + \tau) \cdot h(t) \cdot dt$$

$$S_y(f) = \lambda^2 \cdot |H(0)|^2 \cdot \delta(f) + \lambda \cdot |H(f)|^2$$

Shot Noise – Current Through a p-n Junction



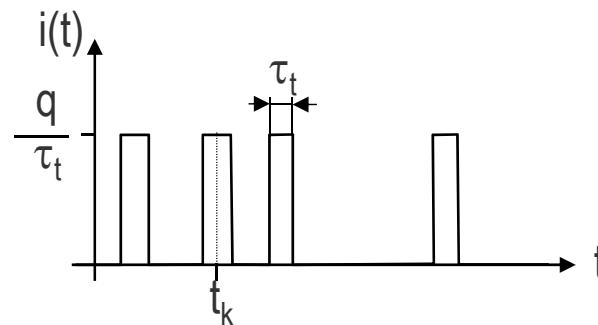
- Neglecting the generation-recombination in the space charge zone

$$I_d = I_s \cdot \left(e^{\frac{V}{U_T}} - 1 \right) \text{ with } U_T \triangleq \frac{kT}{q}$$

- Current I_d is composed of:

- $I_d + I_s = I_s \cdot e^{V/U_T}$: holes injected into the n region recombining there or reaching the ohmic contact
 - $-I_s$: holes generated in the n region and collected by the p region

Shot Noise – Current Through a p-n Junction (cont.)



τ_t : average transit time

$$h(t) = \frac{q}{\tau_t} \cdot \text{rect}\left(\frac{t}{\tau_t}\right)$$

$$\text{with } \text{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

- Assuming that each carrier has the same probability to cross the barrier at any time and that the average number of carrier crossing the barrier per unit time remains constant and equal to λ , the current $i(t)$ is a **shot noise process** with

$$\lambda \cdot q = (I_d + I_s) + I_s = I_d + 2I_s = I_{eq}$$

$$S_I(f) = (\lambda \cdot q)^2 \cdot \delta(f) + \lambda \cdot q^2 \cdot \text{sinc}^2(\pi f \tau_t) = I_{eq}^2 \cdot \delta(f) + q \cdot I_{eq} \cdot \text{sinc}^2(\pi f \tau_t)$$

- The bilateral PSD of the current fluctuation is then given by

$$S_{\Delta I} = q \cdot I_{eq} \cdot \text{sinc}^2(\pi f \tau_t) \cong q \cdot I_{eq} = q \cdot (I_d + 2I_s) \text{ for } f \ll 1/(\pi \tau_t)$$

- Or in unilateral PSD the well-known expression

$$S_{\Delta I} = 2q \cdot I_{eq} \cong 2q \cdot I_d$$



Shot Noise – Current Through a p-n Junction (cont.)

- The unilateral PSD can also be written in terms of the small-signal differential conductance G_d

$$G_d = \frac{dI_d}{dV} = \frac{I_s}{U_T} \cdot e^{\frac{V}{U_T}} = \frac{I_d + I_s}{U_T}$$

- as it would be thermal noise

$$S_{\Delta I} = 2kT \cdot G_d \cdot \frac{I_d + 2I_s}{I_d + I_s}$$

- corresponding to full thermal noise of the conductance $G_{d0} = I_s/U_T$ at zero bias ($V = 0$ and $I_d = 0$)

$$S_{\Delta I} = 4kT \cdot G_{d0}$$

- and “half thermal noise” for $I_d \gg I_s$

$$S_{\Delta I} = 2kT \cdot G_d = 2kT \cdot \frac{I_d}{U_T} = 2qI_d$$



Flicker or 1/f Noise (1/2)

- Random process with zero mean and characterized by its PSD

$$S_{1/f}(f) = \frac{K}{|f|^\alpha}$$

- with α close to one and where K is a constant
- The power (variance) in bandwidth $[f_\ell, f_h]$ assuming $\alpha = 1$ is given by

$$\sigma_{1/f}^2 = 2 \cdot \int_{f_\ell}^{f_h} \frac{K}{f} \cdot df = 2K \cdot \ln\left(\frac{f_h}{f_\ell}\right)$$

- The variance tends to infinity as $f_h \rightarrow \infty$ or $f_\ell \rightarrow 0$
- Divergence of variance at high frequency does not cause a problem since it is always low-pass filtered
- On the other hand, divergence of variance when $f_\ell \rightarrow 0$ causes many controversies and interrogations!



Flicker or 1/f Noise (2/2)

- The “problem” of the divergence of the variance when $f_\ell \rightarrow 0$ can be “solved” when considering the process as **non-stationary**

$$R_{1/f}(t_2, \tau) \approx f(t_2) + f(\tau) \text{ for } 0 < \tau \ll t_2$$

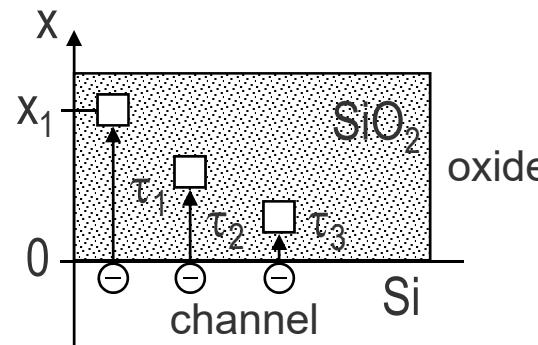
where t_2 is the **age** of the process and $\tau = t_2 - t_1$

- The corresponding PSD follows a 1/f law down to frequency corresponding to the observation time T_{obs} which is independent of the values of t_2 and T_{obs}
- 1/f noise has been observed as fluctuations not only in semiconductors and electronic devices but also in many very different systems, including:
 - ▶ average seasonal temperature,
 - ▶ annual amount of rainfall,
 - ▶ rate of traffic flow,
 - ▶ economic data,
 - ▶ the loudness and pitch of music, etc...



Flicker Noise in the MOST – The Mc-Worther Model (1/3)

cross-section of a MOS transistor showing the mobile carriers (electrons) trapped into the oxide



tunneling time constant

$$\tau_t = \tau_0 \cdot e^{\alpha \cdot x}$$

where $\alpha \approx 10^8 \text{ cm}^{-1}$

- Carrier density fluctuation due to **trapping** via tunneling effect in traps located in the oxide and close to the Si- SiO_2 interface
- Tunneling effect characterized by **tunneling time constant τ_t**
- The PSD corresponding to the fluctuation ΔN in number of electrons (holes) N due to a single trap having a tunneling time constant τ_t is given by a **Lorentzian PSD**

$$S_{\Delta N}(f) = \overline{\Delta N^2} \cdot \frac{2\tau_t}{1 + (2\pi f \tau_t)^2}$$

- where $\overline{\Delta N^2}$ is the variance of ΔN

Flicker Noise in the MOST – The Mc-Worther Model (2/3)

- The variance $\overline{\Delta N^2}$ of ΔN can be assumed to be proportional to N and hence $\overline{\Delta N^2} = \beta \cdot N$ resulting in

$$S_{\Delta N}(f) = \beta \cdot N \cdot \frac{2\tau_t}{1 + (2\pi f \tau_t)^2}$$

- It can be shown that the PSD of the drain current resulting from the charge fluctuation in the channel due to a single trap is given by

$$S_{\Delta I_D}(f) = \left(\frac{I_D}{N}\right)^2 \cdot S_{\Delta N}(f) \cong \frac{\beta \cdot I_D^2}{N} \cdot \frac{2\tau_t}{1 + (2\pi f \tau_t)^2}$$

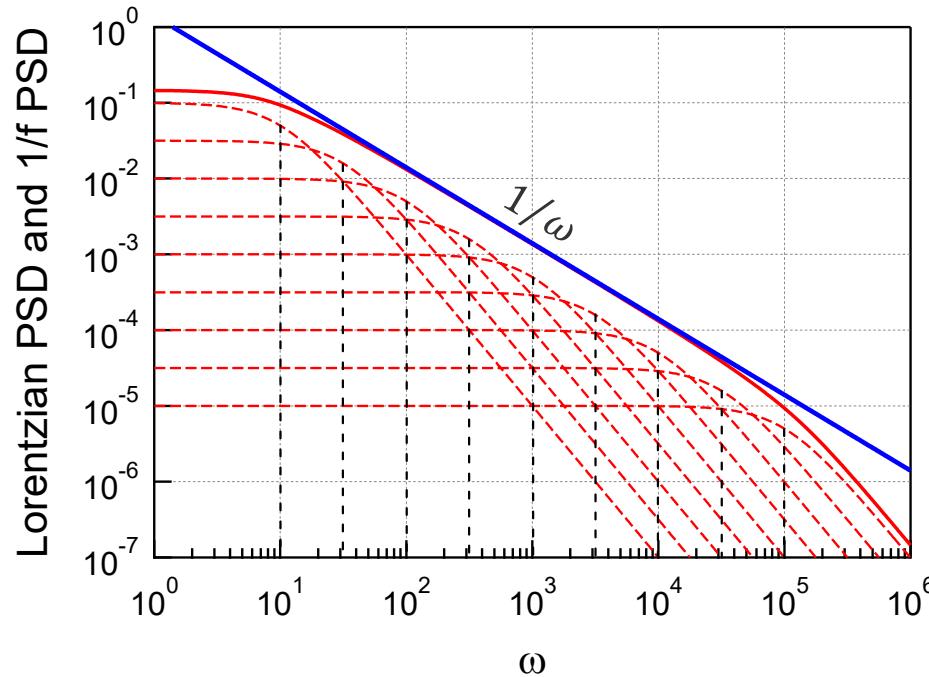
- Averaging over all time constant $\tau_{tmin} \leq \tau_t \leq \tau_{tmax}$ results in a 1/f PSD within $1/(2\pi\tau_{tmax})$ and $1/(2\pi\tau_{tmin})$

$$\frac{S_{\Delta I_D}(f)}{I_D^2} = \frac{K_f}{f}$$

- with

$$K_f = \frac{\beta}{\pi N} \cdot \frac{\text{atan}(2\pi f \tau_{t,max}) - \text{atan}(2\pi f \tau_{t,min})}{\ln(\tau_{t,max}/\tau_{t,min})}$$


Flicker Noise in the MOST – The Mc-Worther Model (3/3)



- The above integration corresponds to summing all the Lorentzian curves corresponding to each trap results in a 1/f power spectrum
- This is illustrated in the above plot with 9 Lorentzians equally spaced (in log scale)

$$S(\omega) = \frac{\tau_t}{1 + (\omega\tau_t)^2} = \frac{1}{\omega_t} \cdot \frac{1}{1 + \left(\frac{\omega}{\omega_t}\right)^2}$$

Flicker Noise in the MOST – The Hooge Model

- Noise due to fluctuations of the **carrier mobility**
- **Volume** effect rather than **interface** effect like in the Mc-Worther model
- The current fluctuation PSD in semiconductor can be expressed as

$$S_{\Delta I}(f) = \frac{\alpha_H}{f \cdot N} \cdot I^2$$

- where $\alpha_H \cong 2 \times 10^{-3}$ is the Hooge parameter, I is the bias current and N the total number of mobile charge
- Applied to the MOS transistor, the flicker noise is often referred to the gate (fluctuations of the gate voltage instead of fluctuations of the drain current)
- In strong inversion and saturation (with $V_S = 0$), this model leads to a PSD

$$S_{\Delta V_G}(f) \cong \frac{\alpha_H \cdot q \cdot (V_G - V_{T0})}{2W \cdot L \cdot C_{ox} \cdot f}$$

- Inversely proportional to gate area $W \cdot L$ and to C_{ox} and proportional to $V_G - V_{T0}$

 F. M. Klaassen, "Characterization of Low 1/f Noise in MOS Transistors," IEEE Transactions on Electron Devices, vol. 18, no. 10, pp. 887-891, Oct. 1971.
 A. van der Ziel, *Noise in Solid-State Devices and Circuits*, Wiley, 1986.

Random Telegraph Noise (RTN)

- CMOS technology scaling led to reduced gate area and lower current levels uncovering the random telegraph noise (RTN)
- Due to trapping and detrapping of mobile charge characterized by very long capturing and emitting time constants τ_c and τ_e
- The PSD of the induced drain current fluctuation due to a single trap is given by

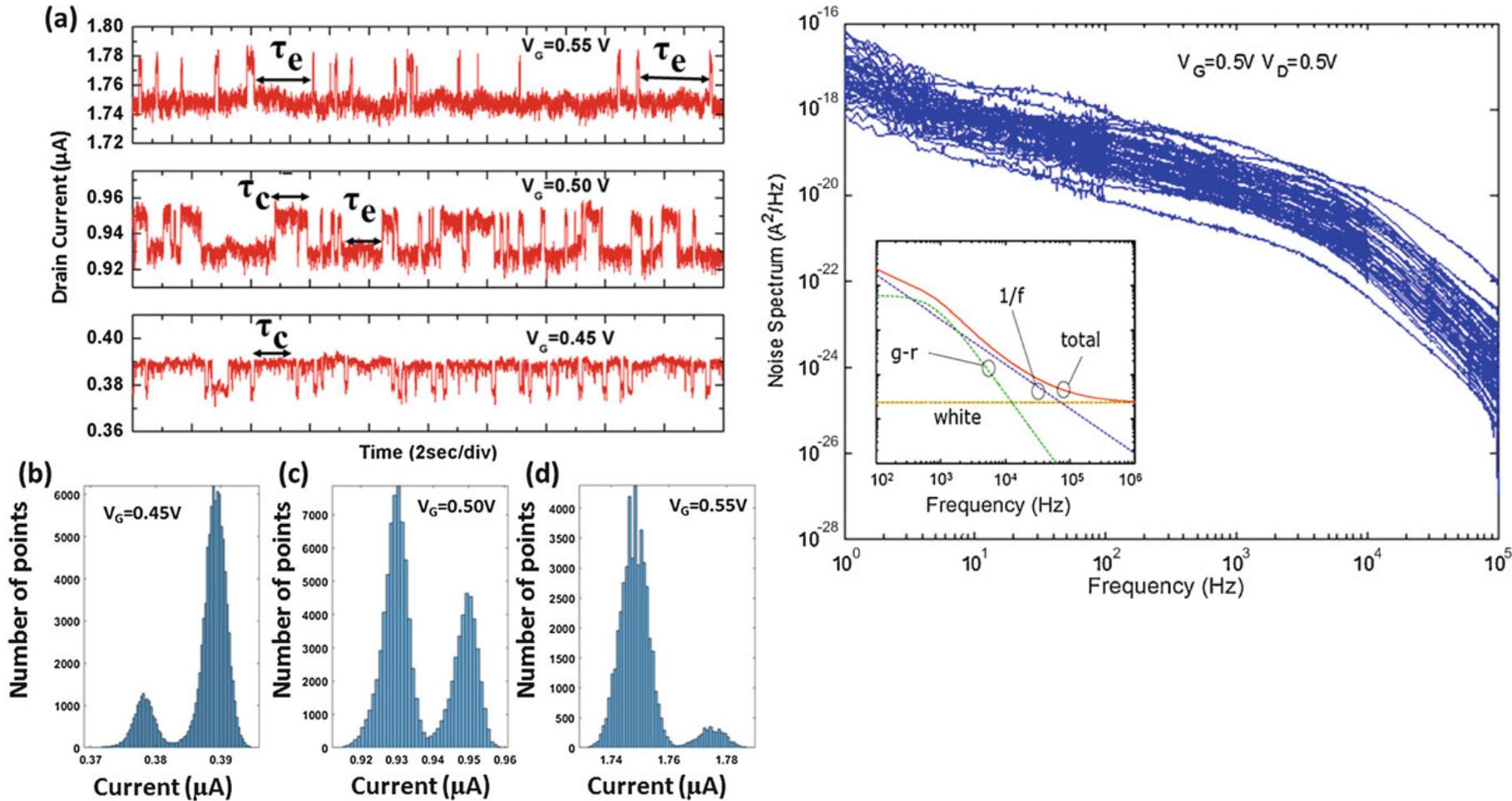
$$\frac{S_{\Delta I_D^2}}{I_D^2} = \frac{\overline{\Delta N^2}}{N^2} \cdot \frac{4\tau_r}{1 + (2\pi f \tau_r)^2}$$

- where $\tau_r = (1/\tau_c + 1/\tau_e)^{-1}$ is the carrier lifetime or effective time constant, $\overline{\Delta N^2}$ is the variance of the fluctuations of the number of carriers in the channel and N is the average number of carriers in the channel
- Note that $\overline{\Delta N^2}$ is proportional to the concentration of generation–recombination-trapping centers
- The PSD of the current fluctuations due to a single trap is then a Lorentzian spectrum given by

$$S_{\Delta I_D^2} = 4 I_D^2 \frac{\tau_r}{\tau_e + \tau_c} \frac{\tau_r}{1 + (2\pi f \tau_r)^2}$$

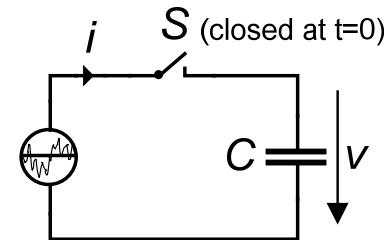
 C. Marquez, et al., "Systematic Characterization of Random Telegraph Noise and Its Dependence with Magnetic Fields in MOSFET Devices," in *Noise in Nanoscale Semiconductor Devices*, ed. T. Grasser (ed.), Springer 2020. https://doi.org/10.1007/978-3-030-37500-3_4.

RTN Time Signature and PSD



C. Marquez, et al., "Systematic Characterization of Random Telegraph Noise and Its Dependence with Magnetic Fields in MOSFET Devices," in *Noise in Nanoscale Semiconductor Devices*, ed. T. Grasser (ed.), Springer 2020. https://doi.org/10.1007/978-3-030-37500-3_4.

Integrated White Noise – Random Walk (Wiener Process) (1/2)



- The integration of a stationary white (current) noise having a PSD and ACF given by

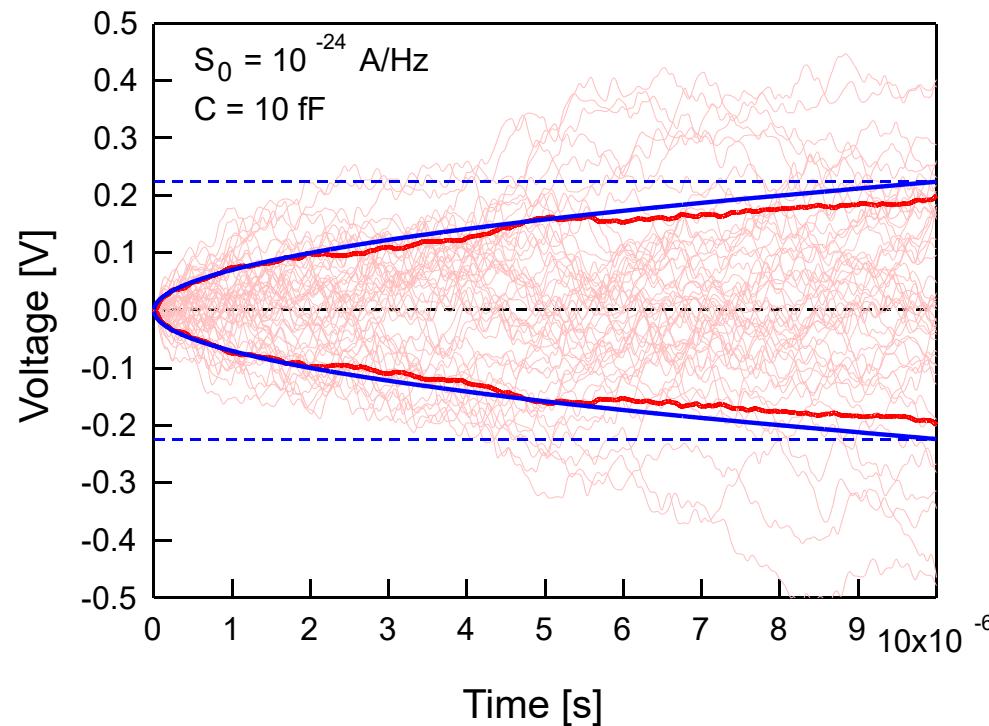
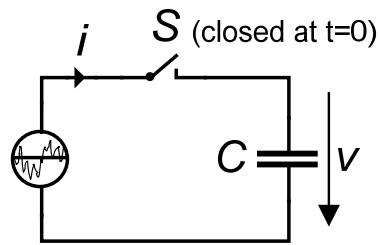
$$S_i(f) = S_0 \text{ and } R_i(\tau) = S_0 \cdot \delta(\tau)$$

- Gives rise to a non-stationary (voltage) noise with an ACF given by

$$R_{vv}(t, \tau) = \begin{cases} \frac{S_0}{C^2} \cdot t & 0 < t < T \\ \frac{S_0}{C^2} \cdot T & 0 < T < t \end{cases}$$

- Note that S_0 is the bilateral FT i.e. if $i(t)$ is thermal noise then $S_0 = 2kT \cdot R$

Integrated White Noise – Random Walk (Wiener Process) (2/2)



- The standard deviation of the voltage (rms value) is therefore increasing with \sqrt{t}
- The above plot shows the standard deviation obtained from transient noise simulations

Other Types of Noise

- Generation-recombination noise
 - ▶ Fluctuation of the conductance due to fluctuating occupancy of the generation-recombination centers
- Quantization noise
 - ▶ Noise introduced by the quantization of analog signal
 - ▶ Not really a random signal, but can be considered as if signal is “busy” and there are many quantization steps
 - ▶ If the quantization error is uniformly distributed over the quantization step, then the variance is given by

$$\sigma_q^2 \approx \frac{\Delta^2}{12} \quad \text{where } \Delta \text{ is the uniform quantization step}$$

- ▶ This noise is signal dependent and the signal-to-noise (SNR) is given by

$$SNR_{dB} \approx 10 \cdot \log(\sigma_x^2) + 6 \cdot b \quad \text{where } \sigma_x^2 \text{ is the signal variance and } b \text{ the number of bits}$$

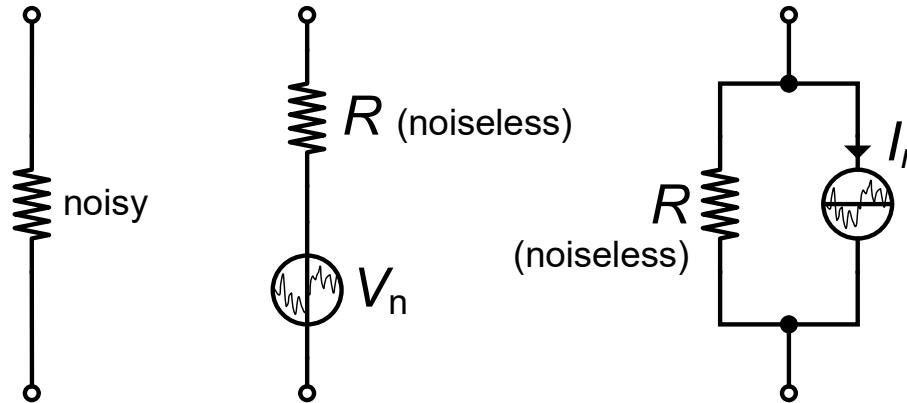
Summary

- **Thermal noise** is due to fluctuation of drift velocity within a conductor and has a PSD **proportional to temperature** and to the **resistance** $S_{th} = 4kT \cdot R$. It is independent of the current flowing through the resistor
- The variance of the noise voltage on the capacitor of a 1st-order low-pass filter is given by kT/C , which is **independent of the resistance** and **inversely proportional to C**
- **Shot noise** is due to carriers randomly crossing a potential barrier. Its PSD is proportional to the average current $S_{sh} = 2q \cdot I$
- **Flicker noise** is due to fluctuation of the number of carrier (Mc Worther model) and/or of the mobility (Hooge model) and has a PSD inversely proportional to frequency
- The flicker noise PSD of MOS transistor is inversely proportional to the **gate area**
- Integrated white noise (Wiener process) has an ACF and a variance which grows with time

Outline

- Introduction
- Random signals and noise
- Main noise sources of circuit components
- **Noise models of basic components**
- Noise calculation in continuous-time (CT) circuits
- Summary

Device Noise – Resistor Noise

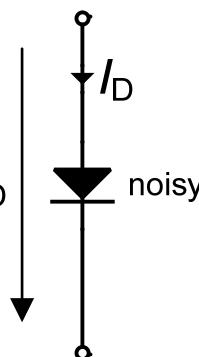


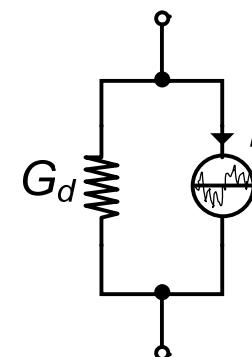
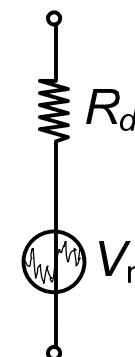
$$S_{V_n} = 4kT \cdot R$$

$$S_{I_n} = \frac{4kT}{R}$$

- **Thermal noise** with k being the Boltzman constant ($k = 1.38 \times 10^{-23} \text{ J/K}$) and T the absolute temperature in K
- Contrary to shot noise, the noise PSD of thermal noise is **independent of the current** flowing through the resistor or voltage across it, it only depends on the resistance R (conductance $G = 1/R$)

Device Noise – Diode Noise

$$I_D = I_S \cdot \left[e^{\frac{V_D}{nU_T}} - 1 \right] \quad V_D$$




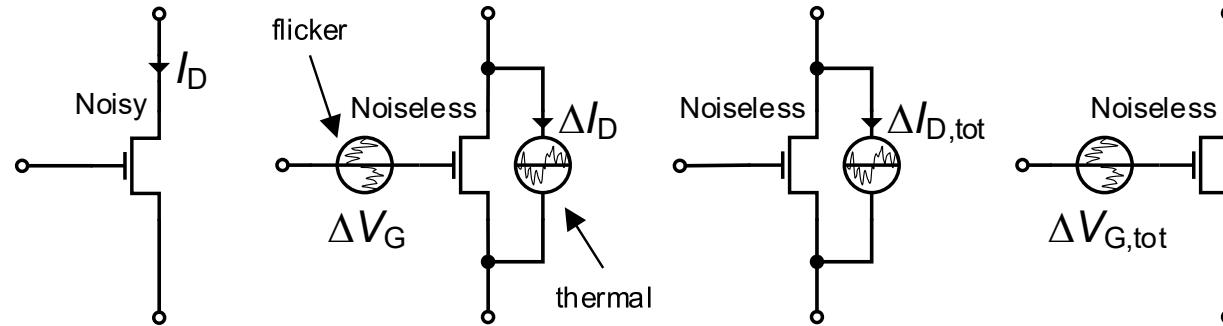
$$G_d = \frac{1}{R_d} \triangleq \frac{dI_D}{dV_D} = \frac{I_D}{nU_T}$$

$$S_{V_n} = 4kT \cdot \frac{R_d}{2}$$

$$S_{I_n} = 4kT \cdot \frac{G_d}{2} = 2q \cdot I_D$$

- Shot noise
- G_d is the small-signal conductance
- Noise PSD **proportional to average current** flowing through the diode

Device Noise – MOSFET (in saturation)



- **Thermal** noise (at the drain):

$$S_{\Delta I_D^2} = 4kT \cdot G_n \text{ with } G_n = \gamma_n \cdot G_m \text{ and } \gamma_n \cong 1 \text{ for long-channel}$$

- **Flicker** noise (at the gate):

$$S_{\Delta V_G^2}(f) = \frac{KF}{C_{ox}^{\alpha} \cdot W \cdot L \cdot f} = 4kT \cdot R_n(f) \text{ with } R_n(f) = \frac{\rho}{W \cdot L \cdot f} \text{ and } \rho = \frac{KF}{4kT \cdot C_{ox}^{\alpha}}$$

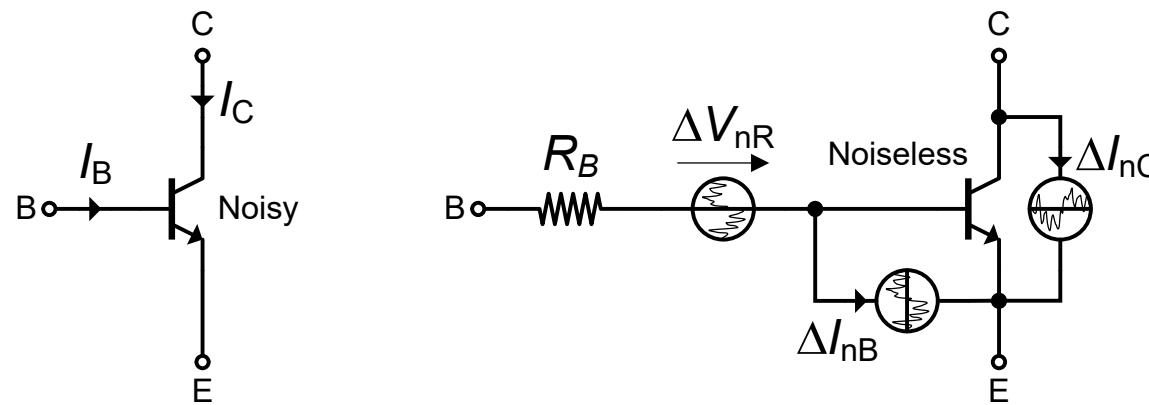
- **Total** noise at the drain:

$$S_{\Delta I_{D,tot}^2}(f) = S_{\Delta I_D^2} + G_m^2 \cdot S_{\Delta V_G^2}(f) = 4kT \cdot G_{n,tot}(f) \text{ with } G_{n,tot}(f) = \gamma_n \cdot G_m + G_m^2 \cdot \frac{\rho}{W \cdot L \cdot f}$$

- **Total** input referred noise (at the gate):

$$S_{\Delta V_{G,tot}^2}(f) = 4kT \cdot R_{n,tot}(f) \text{ with } R_{n,tot}(f) = \frac{\gamma_n}{G_m} + \frac{\rho}{W \cdot L \cdot f}$$

Device Noise – Bipolar Transistor (in forward active)

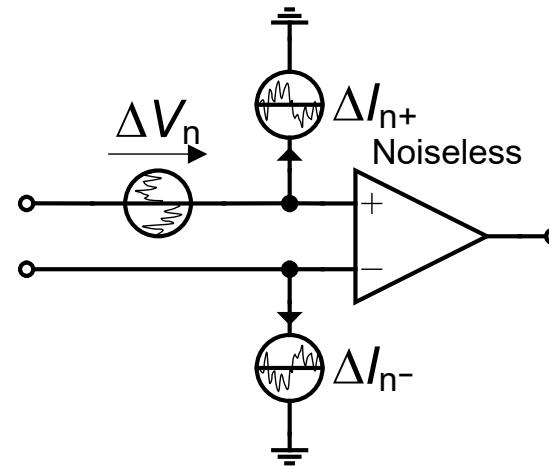
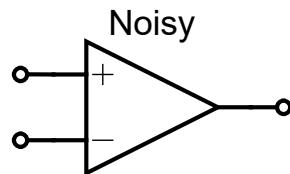


$$S_{\Delta I_{nC}^2} = 2q \cdot I_C$$

$$S_{\Delta I_{nB}^2} = \underbrace{2q \cdot I_B}_{\text{shot noise}} + \underbrace{K_f \cdot \frac{I_B^{a_f}}{f}}_{\text{flicker noise}} + \underbrace{K_p \cdot \frac{I_B^c}{1 + \left(\frac{f}{f_c}\right)^2}}_{\text{burst noise}}$$

$$S_{\Delta V_{nR}^2} = 4kT \cdot R_B$$

Device Noise – OPAMP

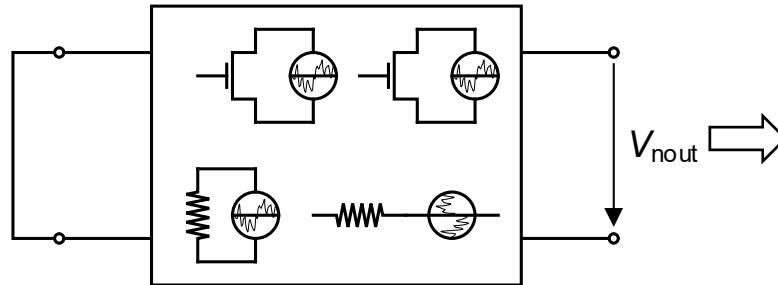


- Requires 3 noise sources
- All 3 noise sources are needed to have a model independent of the generator impedance
- Current noise sources can be ignored for MOS input stage

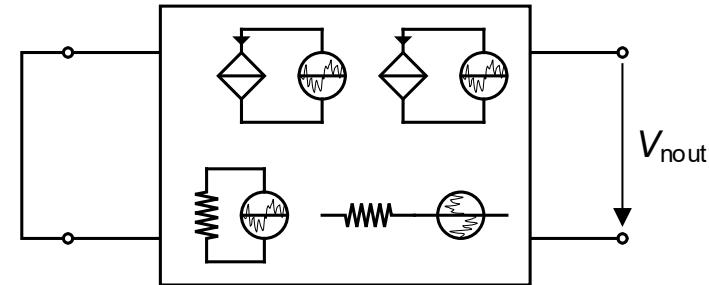
Outline

- Introduction
- Random signals and noise
- Main noise sources of circuit components
- Noise models of basic components
- **Noise calculation in continuous-time (CT) circuits**
- Summary

Noise Analysis in CT Circuits – Small-signal Circuit



Circuit including all noise sources



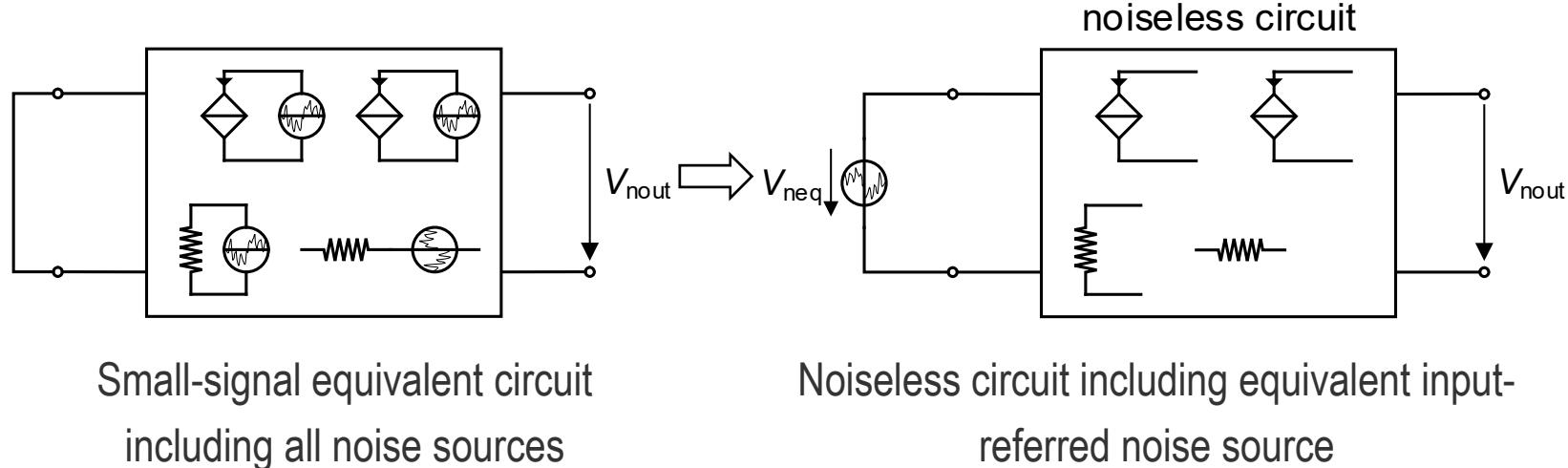
Small-signal equivalent circuit including all noise sources

- Since noise is a small perturbation ($\ll U_T$ at room temperature), the circuit can be **linearized** around an well defined operating point
- The output noise voltage PSD is calculated from the small-signal circuit as

$$S_{nout} = \sum_{k=1}^K |H_k(f)|^2 \cdot S_{nk}(f)$$

- where it has been assumed that all the K noise sources are **uncorrelated**
- H_k are the transfer functions from each noise source k having a PSD S_{nk} to the output

Noise Analysis in CT Circuits – Input-referred Noise

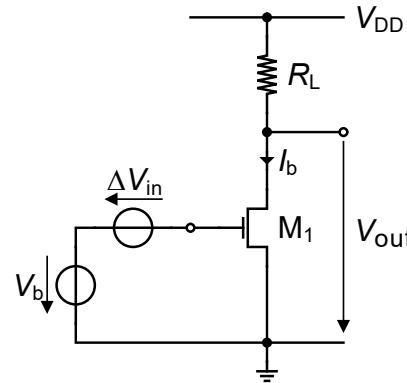


- The noisy circuit can then be modeled by a **noiseless circuit** with a single **input-referred noise** voltage source V_{neq} producing the same noise at the output as that of the noisy circuit having the input short-circuited

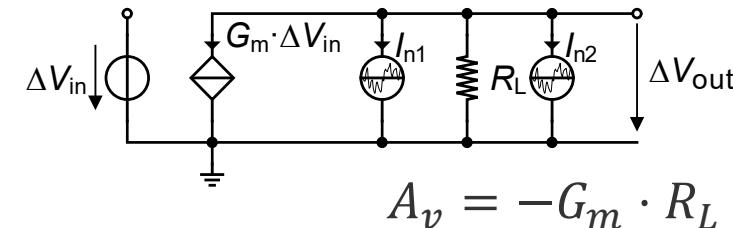
$$S_{V_{\text{neq}}}(f) = \frac{S_{\text{nout}}(f)}{|A(f)|^2}$$

- where $A(f)$ is the transfer function from the input to the output

Example – The CS Gain Stage



Small-signal circuit
including noise sources of M_1 and R_L



- The output noise PSD is given by

$$S_{nout} = R_L^2 \cdot (S_{I_{n1}} + S_{I_{n2}})$$

- where $S_{I_{n1}} = 4 kT \cdot G_{n1}$ and $S_{I_{n2}} = 4 kT \cdot G_{n2}$ with $G_{n1} = \gamma_n \cdot G_m + G_m^2 \cdot \frac{\rho}{W \cdot L \cdot f}$ and $G_{n2} = 1/R_L$

- The input-referred noise PSD is then given by

$$S_{nin} = \frac{S_{nout}}{|A_v|^2} = 4kT \cdot R_{nin}$$

- with $R_{nin} = (G_{n1} + G_{n2})/G_m^2$

Example – The CS Gain Stage

- The input-referred thermal noise resistance is given by

$$R_{nin,th} = \frac{\gamma_n}{G_m} \cdot (1 + \eta_{th})$$

- where the η_{th} factor accounts for the contribution of the load resistance R_L to the input-referred thermal noise

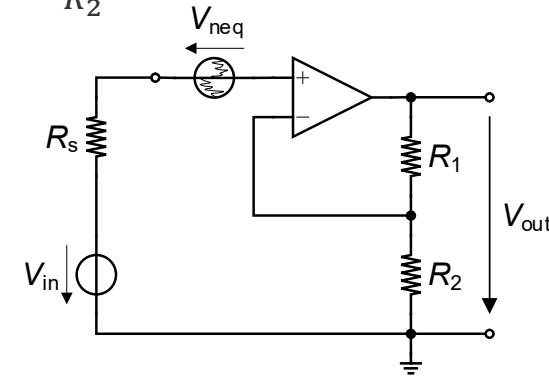
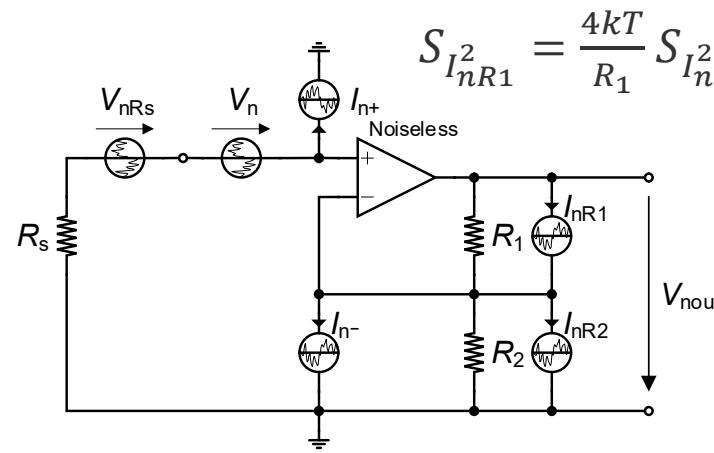
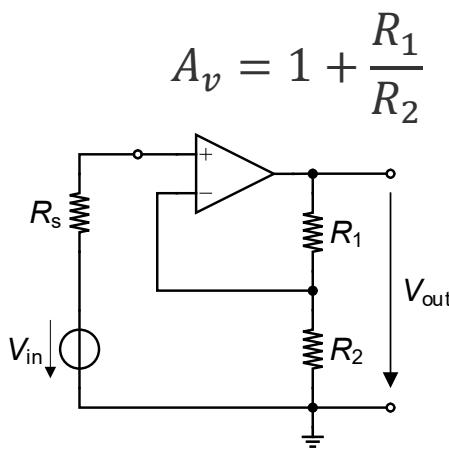
$$\eta_{th} = \frac{1}{\gamma_n \cdot G_m R_L}$$

- We see that the higher the voltage gain $G_m R_L$, the lower the contribution of the load resistance to the input-referred thermal noise
- The input-referred flicker noise resistance is given by

$$R_{nin,fl} = \frac{\rho}{W \cdot L \cdot f}$$

- which only includes the contribution of the transistor M1 since the load resistance is assumed to only generate thermal noise

Example – Noise of the Non-inverting Amplifier

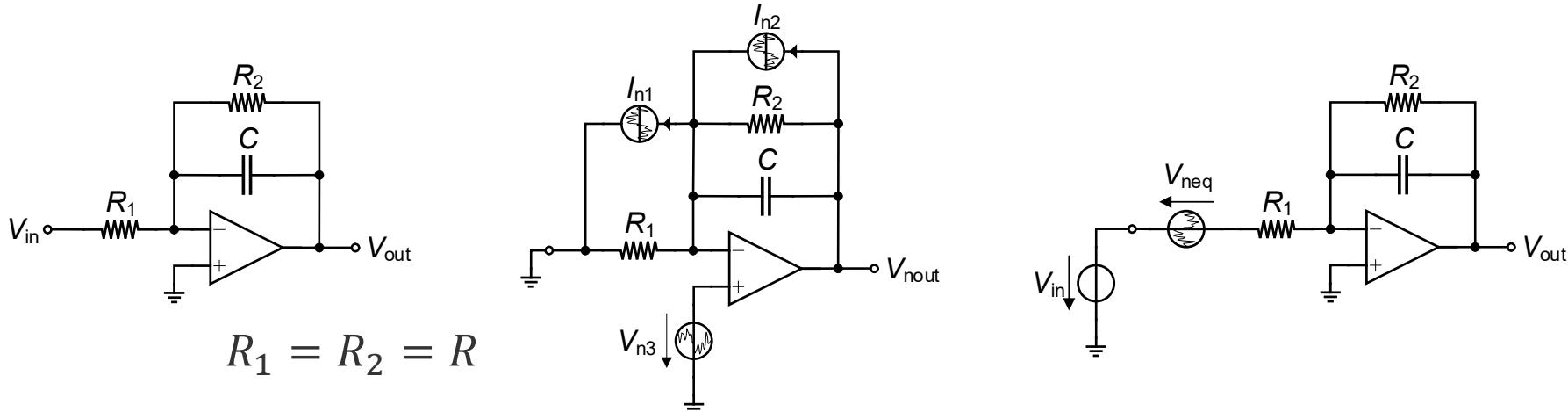


- The non-inverting amplifier has 6 noise source as shown on the middle schematic
- Assuming an ideal amplifier (except the noise) the equivalent input-referred noise voltage PSD is given by

$$S_{V_{neq}} = S_{V_n} + 4kT \cdot (R_s + R_{12}) + R_s^2 \cdot S_{I_{n+}} + R_{12}^2 \cdot S_{I_{n-}} \text{ with } R_{12} = R_1 \parallel R_2$$

- Contribution of R_1 and R_2 equivalent to an additional resistance R_{12} in series with R_s and corresponding to $R_1 \parallel R_2$
- The OPAMP noise current sources start to dominate at high source impedance

Example – 1st-order Low-pass Filter



- Assuming an ideal OPAMP (except the noise where we consider a high input impedance OPAMP and hence the current noise sources can be neglected) and $R_1 = R_2 = R$, the signal transfer function is given by

$$H(s) \triangleq \frac{V_{out}}{V_{in}} = -\frac{1}{1 + \frac{s}{\omega_c}} \text{ with } \omega_c = \frac{1}{R \cdot C} = \frac{1}{\tau}$$

- The output noise voltage is given by

$$V_{nout} = R_{m12} \cdot (I_{n1} - I_{n2}) + H_{n3} \cdot V_{n3}$$

- with

$$R_{m12} = \frac{R}{1 + \frac{s}{\omega_c}} \text{ and } H_{n3} = \frac{2 + \frac{s}{\omega_c}}{1 + \frac{s}{\omega_c}}$$

1st-order CT LPF – Output- and input-referred PSD

- The output noise voltage PSD is then given by

$$S_{V_{nout}}(f) = |R_{m12}(f)|^2 \cdot (S_{I_{n1}} + S_{I_{n2}}) + |H_{n3}(f)|^2 \cdot S_{V_{n3}}$$

- where the noise sources PSD are given by

$$S_{I_{n1}} = S_{I_{n2}} = \frac{4kT}{R} \text{ and } S_{V_{n3}} = 4kT \cdot \frac{\gamma}{G_m} \cdot \left(1 + \frac{f_k}{|f|}\right)$$

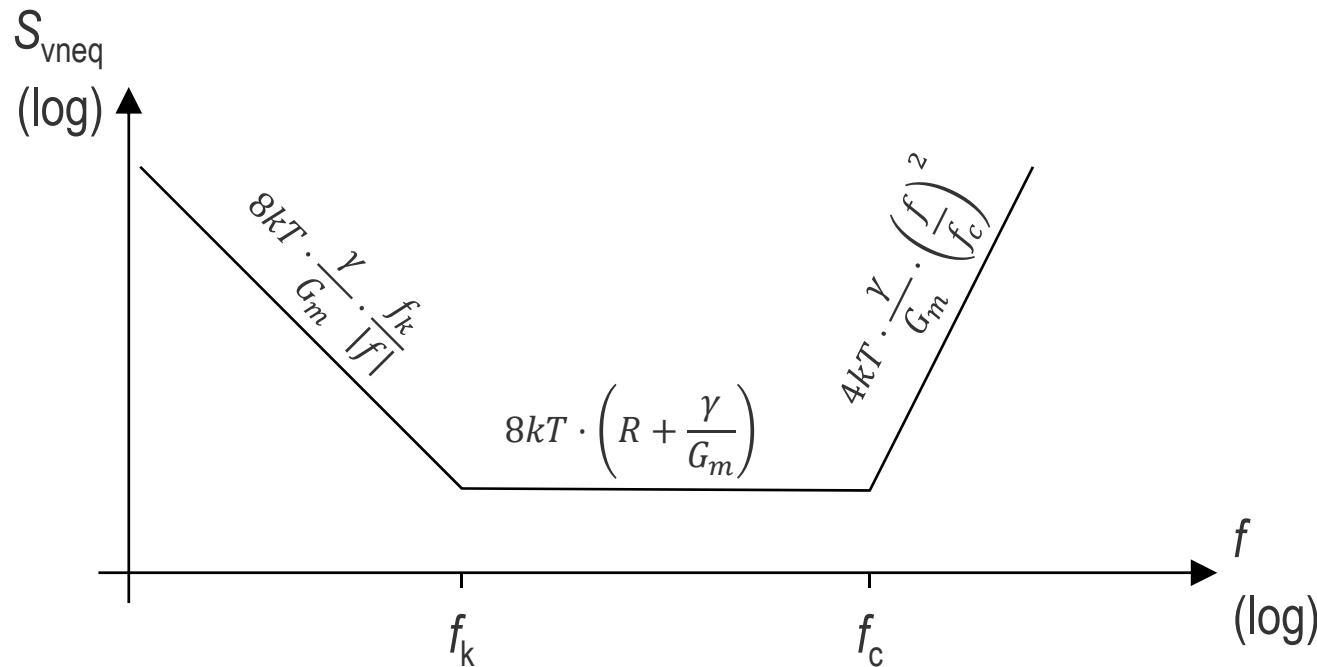
- The output noise voltage PSD is then given by

$$S_{V_{nout}}(f) = \frac{8kT \cdot R}{1 + \left(\frac{f}{f_c}\right)^2} + \frac{2 + \left(\frac{f}{f_c}\right)^2}{1 + \left(\frac{f}{f_c}\right)^2} \cdot 4kT \cdot \frac{\gamma}{G_m} \cdot \left(1 + \frac{f_k}{|f|}\right)$$

- which can be referred to the input as

$$S_{V_{neq}}(f) = 8kT \cdot R + \left[2 + \left(\frac{f}{f_c}\right)^2\right] \cdot 4kT \cdot \frac{\gamma}{G_m} \cdot \left(1 + \frac{f_k}{|f|}\right)$$

1st-order Low-pass Filter – Input Noise Voltage PSD



$$\begin{aligned}
 S_{vneq}(f) &= 8kT \cdot R + \left(2 + \left(\frac{f}{f_c}\right)^2\right) \cdot 4kT \cdot \frac{\gamma}{G_m} \cdot \left(1 + \frac{f_k}{|f|}\right) \\
 &= 8kT \cdot \frac{\gamma}{G_m} \cdot \frac{f_k}{|f|} + 8kT \cdot \left(R + \frac{\gamma}{G_m}\right) + 4kT \cdot \frac{\gamma}{G_m} \cdot \frac{f}{f_c} \cdot \left(\frac{f_k}{|f|} + \frac{f}{f_c}\right) \\
 &\cong 8kT \cdot \frac{\gamma}{G_m} \cdot \frac{f_k}{|f|} + 8kT \cdot \left(R + \frac{\gamma}{G_m}\right) + 4kT \cdot \frac{\gamma}{G_m} \cdot \left(\frac{f}{f_c}\right)^2 \text{ since } f_k < f_c
 \end{aligned}$$

1st-order CT LPF – Output Noise Voltage Variance

- The variance of the output noise voltage can be calculated by integrating the output noise voltage PSD over frequency
- The contribution of the resistor R_1 and R_2 are 1st-order low-pass filtered
- The corresponding noise bandwidth is then simply

$$B_{n12} = \frac{\pi}{2} \cdot f_c = \frac{\pi}{2} \cdot \frac{1}{2\pi\tau} = \frac{1}{4\tau} = \frac{1}{4RC}$$

- The variance of the output noise voltage due to R_1 and R_2 is then simply given by

$$V_{nout}^2 \Big|_{R_1, R_2} = \frac{8kT}{R} \cdot |R_{m12}(0)|^2 \cdot B_{n12} = \frac{8kT}{R} \cdot R^2 \cdot \frac{1}{4RC} = \frac{2kT}{C}$$

- The contribution of V_{n3} to the output noise voltage PSD is not frequency bounded and hence its contribution to the output noise voltage variance is infinite
- In order to evaluate the output noise voltage variance due to V_{n3} , we need to account for the amplifier finite bandwidth

1st-order CT LPF – Amplifier Noise Transfer Function

- If the amplifier gain is assumed to be A , then the transfer function H_{n3} becomes

$$H_{n3}(s) = \frac{2 + \frac{s}{\omega_c}}{1 + \frac{2}{A} + \left(1 + \frac{1}{A}\right) \frac{s}{\omega_c}}$$

- Note that for $A \rightarrow \infty$, we recover the earlier result
- If A is simply approximated by $A = \omega_u/s$ where $\omega_u = G_m / C_c$ is the **gain-bandwidth product** with C_c being the **compensation capacitance**, H_{n3} becomes

$$H_{n3}(s) = \frac{2 + \frac{s}{\omega_c}}{1 + \left(2 + \frac{\omega_u}{\omega_c}\right) \frac{s}{\omega_c} + \frac{s^2}{\omega_u \omega_c}} = H_{n3}(0) \cdot \frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2}$$

- with $H_{n3}(0) = 2$, $\omega_z = 2\omega_c$, $\omega_0^2 = \omega_c \cdot \omega_u$, $\omega_0 \cdot Q = \omega_u / (2 + \omega_u / \omega_c)$
- The amplifier noise is now bounded by the frequency dependence of the amplifier gain

1st-order CT LPF – Total Output Noise Voltage Variance

- From the table, it can be shown that the noise bandwidth corresponding to the 2nd-order transfer function H_{n3} is given by

$$B_{n3} = \frac{\omega_0 \cdot Q}{4} \cdot \left[1 + \left(\frac{\omega_0}{\omega_z} \right)^2 \right] = \frac{\omega_u}{16} \cdot \frac{4 + \omega_u/\omega_c}{2 + \omega_u/\omega_c} \cong \frac{\omega_u}{16} \text{ for } \omega_c \ll \omega_u$$

- The output noise voltage variance due to V_{n3} is then given by

$$V_{nout}^2 \Big|_{V_{n3}} = 4kT \cdot \frac{\gamma}{G_m} \cdot |H_{n3}(0)|^2 \cdot B_{n3} \cong 4kT \cdot \frac{\gamma}{G_m} \cdot 4 \cdot \frac{\omega_u}{16} = \frac{\gamma kT}{C_c}$$

- The total output noise voltage variance is then given by

$$V_{nout}^2 \cong \frac{2kT}{C} + \frac{\gamma kT}{C_c} = \frac{kT}{C_c} \cdot \left(\gamma + \frac{C_c}{C} \right) \cong \frac{\gamma kT}{C_c} \text{ for } C_c \ll C$$

- Since the voltage gain in the passband is unity, this also corresponds to the total input-referred noise voltage variance
- Note that if $C_c \ll C$, the noise voltage variance at the output is then mainly determined by the OPAMP thermal noise excess factor γ and its compensation capacitor C_c !

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Summary

- Noise is a random source of perturbation inherent to any passive and active electronic device
- Because it is random it can be reduced by different techniques (such as filtering), but can never be eliminated
- Thermal noise has a PSD given by $S_{th} = 4kT \cdot R$, which is independent of the current flowing through the resistor
- The variance of the noise voltage on the capacitor of a 1st-order low-pass filter is given by kT/C
- Shot noise has a PSD proportional to the average current $S_{sh} = 2q \cdot I$
- The gate-referred thermal noise PSD of a MOS transistor is given by $S_{\Delta V_G^2} = 4kT \cdot \gamma_{nD}/G_m$
- The gate-referred flicker noise PSD of a MOS transistor is almost biased independent and given by $S_{\Delta V_G^2} = KF/(C_{ox}^\alpha \cdot W \cdot L \cdot f)$